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4 SEM TDC MTMH (CBCS) C 9

2022

(June/July)

MATHEMATICS

(Core)

Paper : C–9

(Riemann Integration and Series of Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* State two partitions of the interval [1, 2] such that one is a refinement of the other. 1

(b) Consider the function $f(x) = x$ on [0, 1] and the partitions

$$P = \{x_i = \frac{i}{4},\ i = 0, 1, 2, 3, 4\}$$

$$Q = \{x_j = \frac{j}{4},\ j = 0, 1, 2, 3, 4, 5, 6\}$$

Determine the lower sums and upper sums of f with respect to P and Q. State the relations between $L(f, P)$ and $L(f, Q)$; $U(f, P)$ and $U(f, Q)$. 4

Or

For a bounded function f on $[a, b]$ with its bounds m and M, show that

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a)$$

for any partition P of $[a, b]$.

2. *(a)* Define a tagged portion of a closed interval. Define Riemann sum of a bounded function. 1+1=2

(b) Let $f : [a, b] \to \mathbb{R}$ be integrable. Then show that

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

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(c) Answer any *four* questions from the following : 5×4=20

(i) Let $f : [a, b] \to \mathbb{R}$ be bounded and monotonic. Then show that f is integrable.

(ii) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then show that f is integrable.

(iii) Let $f : [a, b] \to \mathbb{R}$ be integrable. Define F on $[a, b]$ as $F(x) = \int_a^x f(t)dt$; $x \in [a, b]$. Show that F is continuous on $[a, b]$.

(iv) Let f be continuous on $[a, b]$. Show that there exists $c \in [a, b]$ such that

$$\frac{1}{b-a} \int_a^b f(x)dx = f(c)$$

(v) Show that if $f : [a, b] \to \mathbb{R}$ is integrable, then $|f|$ is integrable on $[a, b]$.

(vi) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then show that f is bounded on $[a, b]$.

3. (a) Discuss the convergence of $\int_1^\infty \frac{dx}{x^p}$ for various values of p. 3

(b) Attempt any *one* :

Show that—

(i) $B(m, n) = B(n, m)$

(ii) $\Gamma(m+1) = \underline{|m}$; $m \in \mathbb{N}$ 3

(c) Show that $\int_0^\infty x^{n-1} e^{-x} dx$ exists. 4

4. (a) Define pointwise convergence of sequence of functions. 1

(b) Define uniform convergence of sequence of functions. 2

(c) State and prove Weierstrass *M*-test for the series of functions. 4

(d) State and prove Cauchy's criterion for uniform convergence of a series of functions. 4

Or

Let $f_n : J \subseteq \mathbb{R} \to \mathbb{R}$ converge uniformly on J to f. Let $f_n \; \forall n$ is continuous at $a \in J$. Then show that f is continuous at a.

(e) Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ and $f_n \to f$ uniformly on $[a, b]$. Show that f is continuous and therefore integrable. Establish that

$$\int_a^b f(x)dx = \lim \int_a^b f_n(x)dx$$ 4

(f) Let $f_n : (a, b) \to \mathbb{R}$ be differentiable. Let there exist functions f and g defined on (a, b) such that $f_n \to f$ and $f_n' \to g$ uniformly on (a, b). Show that f is differentiable and $f' = g$ on (a, b). 5

(g) Consider the function $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{\sin nx}{n}$. Show that (f_n) converges pointwise and uniformly to the zero function. 5

5. (a) Define a power series around a real number c. Give an example of power series around the origin. 1+1=2

(b) Define radius of convergence of a power series. Show that the radius of convergence R of a power series $\sum a_n x^n$ is given by $\dfrac{1}{R} = \lim \left| \dfrac{a_{n+1}}{a_n} \right|$. 4

(c) State and prove Cauchy-Hadamard theorem. 4

(d) Show that if the series $\sum a_n$ converges, then the power series $\sum a_n x^n$ converges uniformly on $[0, 1]$. 5

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