

Total No. of Printed Pages—7

3 SEM TDC MTMH (CBCS) C 5

2023

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C-5

(Theory of Real Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

1. (a) Give an example of a proper subset of \mathbb{R} , whose cluster points are the elements of the proper subset itself. 1
- (b) State whether true or false : 1

In the definition of $\lim_{x \rightarrow c} f$ where c is a cluster point of the domain of f , it is immaterial whether f is defined at c or not.

- (c) Use the definition of limit of a function to show that

$$\lim_{x \rightarrow c} x = c \quad 1$$

- (d) Use ϵ - δ definition to establish that

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0 \quad 2$$

- (e) Let $f : A \rightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$ and c is a cluster point of A . Show that if $\lim_{x \rightarrow c} f(x) = L$, then $\lim_{x \rightarrow c} |f(x) - L| = 0$. 2

- (f) Define a bounded function with a suitable example. 2

- (g) Let $f : A \rightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$ and c is a cluster point of A . If $\lim_{x \rightarrow c} f < 0$, then show that there exists a neighbourhood $V_\delta(c)$ of c such that $\forall x \in A \cap V_\delta(c)$ with $x \neq c$, $f(x) < 0$. 3

Or

Use squeeze theorem to show that

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

- (h) Let $f : A \rightarrow \mathbb{R}$ and $g : B \rightarrow \mathbb{R}$ be functions where $A, B \subseteq \mathbb{R}$ and $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $b = f(c) \in B$, then show that the composition $gf : A \rightarrow \mathbb{R}$ is continuous at $c \in A$. 3
- (i) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b] \subseteq \mathbb{R}$. Then show that f is bounded on $[a, b]$. 3
- (j) Let $f : I \rightarrow \mathbb{R}$ be continuous on I , an interval. If $a, b \in I$ and $k \in \mathbb{R}$ satisfy $f(a) < k < f(b)$, then show that there exists a point $c \in I$ between a and b such that $f(c) = k$. 4
- (k) Let $f : I \rightarrow \mathbb{R}$ be continuous where I is a closed bounded interval in \mathbb{R} . Show that the set $f(I) = \{f(x) : x \in I\}$ is a closed bounded interval. 4
- (l) Use sequential criteria of continuity to establish that Dirichlet's function is not continuous at any real numbers. 4

2. (a) State Caratheodory's theorem. 1

(b) State whether true or false : 1

Let x_0 be an interior point of an interval I and the derivatives $f', f'', \dots, f^{(n)}$ exist and continuous in a neighbourhood of x_0 with $f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. If n is odd, then f has no relative extremum at x_0 .

(c) Use first derivative test to establish that f defined by $f(x) = x^3$ has no extremum at $x = 0$. 2

(d) Find the relative extremum of the function $f(x) = \sum_{i=1}^n (a_i - x)^2$ where $a_i \in \mathbb{R}$; $1 \leq i \leq n$. 2

(e) Let f be continuous on an interval $[a, b]$ and differentiable on (a, c) and (c, b) where $c \in (a, b)$. Then if there exists a neighbourhood $(c - \delta, c + \delta)$ of c in $[a, b]$ such that $f'(x) \geq 0 \forall x \in (c - \delta, c)$ and $f'(x) \leq 0 \forall x \in (c, c + \delta)$, then show that f has a relative maximum at c . 3

Or

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$.

Let $c \in I$ and $f'(c)$ exist and positive.

Then show that there exists a number $\delta > 0$ such that $f(x) > f(c)$

$\forall x \in (c, c + \delta)$.

- (f) Let f be continuous on a closed interval $[a, b]$ and differentiable on an open interval (a, b) . If $f'(x) = 0 \forall x \in (a, b)$, show that f is constant on $[a, b]$. Hence, show that if g is another function satisfying that it is continuous on $[a, b]$ and differentiable on (a, b) with $f'(x) = g'(x) \forall x \in (a, b)$, then there exists a constant k such that $f(x) = g(x) + k$ on $[a, b]$. 4
- (g) Use mean value theorem to show that $\frac{x-1}{x} < \log x < x-1$ for $x > 1$. 4
- (h) State and prove Rolle's theorem and give its geometrical interpretation. 4
- (i) Use Taylor's theorem to show that

$$1 - \frac{x^2}{2} \leq \cos x \forall x \in \mathbb{R} \quad 4$$

Or

Show that if $\alpha > 1$, then

$$(1+x)^\alpha > 1 + \alpha x \quad \forall x > -1$$

and $x \neq 0$.

3. (a) Consider Cauchy's mean value theorem for two functions f and g which are continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0 \quad \forall x \in (a, b)$. For what value of $g(x)$, Cauchy's mean value theorem reduces to mean value theorem? 1
- (b) State Lagrange's form of remainder in Taylor's theorem for a function f defined on $[a, b]$. 1
- (c) State Maclaurin's infinite series expansion about $x = 0$ mentioning the interval of expansion. 2
- (d) Investigate whether the function $f : (0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = x \log x$ is convex or not. 2
- (e) Investigate the function
$$f(x) = (x-3)^5(x+1)^4$$
 for relative extrema. 4
- (f) Let $f : I \rightarrow \mathbb{R}$ have second derivative on an open interval I of \mathbb{R} . Show that f is a convex function on I if and only if $f''(x) \geq 0 \quad \forall x \in I$. 5

(7)

(g) Expand $\cos x$ in the Maclaurin's series. 5

Or

Expand $\log(1+x)$ in the Maclaurin's series.

(h) State and prove Taylor's theorem with Cauchy's remainder. 5
