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6 SEM TDC MTMH (CBCS) C 13

2024

(May)

MATHEMATICS

(Core)

Paper : C–13

(Metric Spaces and Complex Analysis)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Write the triangle inequality of metric space. 1

(b) A metric d on a non-empty set may be negative. State True or False. 1

(c) A metric space consists of two objects. Write that objects. 2

(d) Define a pseudometric on a non-empty set. 2

(e) Define a complete metric space. 2

(f) Answer any *two* from the following : 6×2=12

(i) Show that in any metric space X, each open sphere is an open set.

(ii) Let X be a metric space with metric d. Show that d_1 defined by

$$d_1 = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X.

(iii) Show that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

(iv) Show that a subset of a metric space is bounded if and only if it is non-empty and is contained in some closed sphere.

2. (a) Write when a metric space is called sequentially compact. 1

(b) Write an example of an uniformly continuous function in a metric space. 1

(c) Define a continuous mapping in a metric space. 2

(d) Show that the homeomorphism on the set of all metric spaces is an equivalence relation. 5

Or

Let f is a continuous mapping of a metric space X into a metric space Y. Then show that if E is a connected subset of X, then $f(E)$ is connected.

(e) Let X and Y be metric spaces and f a mapping of X into Y. Then show that f is continuous at x_0 if and only if $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$. 6

Or

Show that every compact metric space has the Bolzano-Weierstrass property.

3. (a) Define extended complex plane. 1

(b) If a function f is continuous throughout a region R, then it is not bounded. State True or False. 1

(c) Show that $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist. 2

(d) Find the arg z, where $z = \frac{-5}{1+i\sqrt{2}}$. 2

(e) Show that $\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$, $w = w(r, \theta)$ is an analytic function. 4

Or

Let $f(z) = z - \bar{z}$. Show that $f'(z)$ does not exist at any point.

(f) Describe the mapping $w = z^2$. 5

4. (a) Find the analytic function $f(z) = u + iv$, where $u(x, y) = \sinh x \sin y$. 5

(b) e^z may have negative value. State True or False. 1

(c) Show that $\log(e^z) = z + 2n\pi i,\ n = 0, 1, 2, \cdots$. 4

Or

Evaluate $\int_C \bar{z}\, dz$, where C is the right-hand half of the circle $|z| = 2$.

5. (a) If a series of complex numbers converges, then write to which the nth term converges as n tends to infinity. 1

(b) Find the limit to which the sequence $z_n = \frac{1}{n^3} + i,\ n = 1, 2, \cdots$ converges. 2

(c) State and prove Liouville's theorem. 7

Or

Find the Taylor's series for the function $\frac{1}{(1+z^2)(z+2)}$, when $|z| < 1$.

6. (a) Define absolute convergence of a power series. 2

(b) Define the circle of convergence of a power series. 2

(c) Write when a power series is called uniformly convergent. 1

(d) Find Laurent's series for the function

$$f(z) = \frac{4z+3}{z(z-3)(z+2)}$$

when $2 < |z| < 3$. 5

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