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**6 SEM TDC MTMH (CBCS) C 14**

**2024**

( May )

MATHEMATICS

( Core )

Paper : C–14

**( Ring Theory and Linear Algebra—II )**

*Full Marks* : 80

*Pass Marks* : 32

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

**1.** Answer any *three* from the following : 5×3=15

(a) Define polynomial ring and prove that if $D$ is an integral domain, then $D[x]$ is also an integral domain.

(b) Let $F$ be a field. Then prove that $F[x]$ is principal ideal domain.

(c) State division algorithm for $F[x]$ and find the quotient and remainder upon dividing $f(x) = 3x^4 + x^3 + 2x^2 + 1$ by $g(x) = x^2 + 4x + 2$ where $f(x)$ and $g(x)$ belong to $Z_5[x]$.

(d) State Eisenstein's criterion on irreducibility. And prove that, in an integral domain, every prime is irreducible.

**2.** Answer any *three* from the following : 5×3=15

(a) Prove that if $F$ is a field, then $F[x]$ is Euclidean domain.

(b) Prove that every ideal of Euclidean domain is principal ideal.

(c) Prove that every principal ideal domain is unique factorization domain.

(d) Show that the ring

$$Z[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a,\ b \in Z\}$$

is an integral domain but not a unique factorization domain.

**3.** Answer any *three* from the following : 6×3=18

*(a)* Let $V$ be an $n$-dimensional vector space over the field $F$ and let $B = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis for $V$. Let $B' = \{f_1, f_2, ..., f_n\}$ be the dual basis of $B$. Then prove that—

*(i)* for each linear functional $f$ on $V$,

$$f = \sum_{i=1}^{n} f(\alpha_i) f_i;$$

*(ii)* for each vector $\alpha$ in $V$, $\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$.

*(b)* Find the dual basis of the basis set

$$B = \{(1, -2, 3), (1, -1, 1), (2, -4, 7)\}$$

of $V_3(R)$.

*(c)* Let $W_1$ and $W_2$ be subspaces of a finite dimensional vector space $V$. Then prove that—

*(i)* $(W_1 + W_2)^\circ = W_1^\circ \cap W_2^\circ$;

*(ii)* $(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$.

*(d)* Find the minimal polynomial of the real matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also, show that minimal polynomial of a matrix or of a linear operator is unique.

**4.** (a) Show that the space generated by (1, 1, 1) and (1, 2, 1) is an invariant subspace of $R^3$ under $T$, where

$$T(x, y, z) = (x+y-z, x+y, x+y-z)$$ 3

(b) Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable over the field $C$. 4

*Or*

Find all complex eigenvalues and eigen-spaces of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

**5.** (a) If $\alpha$ and $\beta$ are vectors in an inner product space $V(F)$ and $a, b \in F$, then prove that—

(i) $\|a\alpha + b\beta\|^2 = |a|^2\|\alpha\|^2 + a\bar{b}(\alpha, \beta) + \bar{a}b(\beta, \alpha) + |b|^2\|\beta\|^2$;

(ii) $\text{Re}(\alpha, \beta) = \frac{1}{4}\|\alpha+\beta\|^2 - \frac{1}{4}\|\alpha-\beta\|^2$. 5

(b) If $\alpha$ and $\beta$ are vectors in a real inner product space and if $\|\alpha\| = \|\beta\|$, then prove that $\alpha - \beta$ and $\alpha + \beta$ are orthogonal and interpret the result geometrically.

3+2=5

(c) Given the basis (2, 0, 1), (3, −1, 5) and (0, 4, 2) for $V_3(R)$. Construct from it by the Gram-Schmidt process an orthonormal basis relative to the standard inner product space. 5

*Or*

Let $V$ be a finite dimensional inner product space and let $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be an orthonormal basis for $V$. Show that for any vectors $\alpha, \beta \in V$,

$$(\alpha, \beta) = \sum_{k=1}^{n} (\alpha, \alpha_k)\overline{(\beta, \alpha_k)}$$

**6.** (a) If $T$ is skew, does it follow that so is $T^2$? What about $T^3$? 1+1=2

(b) Answer any *two* from the following : 4×2=8

(i) Let $V$ be the vector space $V_2(C)$ with the standard inner product. Let $T$ be the linear operator defined by

$$T(1, 0) = (1, -2), \quad T(0, 1) = (i, -1)$$

If $\alpha = (a, b)$, then find $T^*\alpha$.

*(ii)* Prove that a linear transformation $E$ is an orthogonal projection if and only if $E = E^2 = E^*$.

*(iii)* Prove that a necessary and sufficient condition that a self-adjoint linear transformation $T$ on an inner product space $V$ be $\hat{0}$ is that $(T\alpha, \alpha) = 0, \forall \alpha \in V$.

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