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**2 SEM TDC MTMH (CBCS) C 3**

**2 0 2 2**

( June/July )

**MATHEMATICS**

( Core )

Paper : C-3

( **Real Analysis** )

Full Marks : 80

Pass Marks : 32

Time : 3 hours

*The figures in the margin indicate full marks  
for the questions*

1. (a) Define  $\varepsilon$ -neighbourhood of a point. 1
- (b) Find the infimum and supremum, if it exists for the set  $A = \{x \in \mathbb{R} : 2x + 5 > 0\}$ . 2

( 2 )

(c) If

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

then show that  $\inf S = 0$ , where  $\inf S$  denotes the infimum of  $S$ .

3

(d) State and prove that Archimedean Property of real numbers.

4

(e) Let  $S \subseteq \mathbb{R}$  be a set that is bounded above and for  $a \in \mathbb{R}$ ,  $a + S$  is defined as  $a + S = \{a + s : s \in S\}$ . Show that  $\sup(a + S) = a + \sup(S)$ , where  $\sup(S)$  denotes the supremum of  $S$ .

5

2. (a) State the Completeness Property of real numbers.

1

(b) Show that

$$\sup \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\} = 1$$

2

(c) Let

$$I_n = \left[ 0, \frac{1}{n} \right]$$

for  $n \in \mathbb{N}$ . Prove that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$

3



( 3 )

- (d) Prove that the set of real numbers is not countable.

4

Or

If

$$S = \left\{ \frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N} \right\}$$

find  $\inf S$  and  $\sup S$ .

- (e) State and prove the nested interval property.

5

Or

Prove that there exists a real number  $x$  such that  $x^2 = 2$ .

3. (a) State the Monotone Subsequence Theorem.

1

- (b) Show that

$$\lim_{n \rightarrow \infty} \left( \frac{n}{n^2 + 1} \right) = 0$$

2

- (c) Show that a convergent sequence of real numbers is bounded.

3

(d) Show that

$$\lim_{n \rightarrow \infty} (b^n) = 0$$

if  $0 < b < 1$ .

4

Or

Show that

$$\lim_{n \rightarrow \infty} (c^{\frac{1}{n}}) = 1$$

for  $c > 1$ .

(e) State and prove the Monotone Convergence theorem.

5

Or

Let  $Y := (y_n)$  be defined as  $y_1 = 1$ ,  $y_{n+1} = \frac{1}{4}y_n + 2$ ,  $n \geq 1$ . Show that  $(y_n)$  is monotone and bounded. Find the limit.

4. (a) Give an example of two divergent sequences such that their sum converges.

1

(b) Prove that the limit of a sequence of real numbers is unique.

2



(c) Prove that

$$\lim_{n \rightarrow \infty} x_n = 0$$

if and only if

$$\lim_{n \rightarrow \infty} (|x_n|) = 0 \quad 3$$

(d) Establish the convergence or divergence of the following sequences (any one) : 4

$$(i) \ x_n = \frac{(-1)^n n}{n+1}$$

$$(ii) \ x_n = \frac{n^2}{n+1}$$

$$(iii) \ x_n = \frac{2n^2 + 3}{n^2 + 1}$$

(e) Define Cauchy sequence. Prove that a sequence of real numbers is Cauchy if and only if it is convergent. 1+4=5

Or

Establish the convergence or divergence of the sequence

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$$

for  $n \in \mathbb{N}$ .

5

5. (a) State the Cauchy Criterion for convergence of a series. 1

- (b) Prove that if

$$\sum_{n=1}^{\infty} x_n$$

converges then

$$\lim_{n \rightarrow \infty} (x_n) = 0 \quad 3$$

- (c) Prove that if

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. 3

- (d) Show that the series

$$\sum_{n=1}^{\infty} x_n$$

converges if and only if the sequence  $S = (s_k)$  of partial sums is bounded. 4

- (e) Define absolute convergence. Show that if a series of real numbers is absolutely convergent then it is convergent. 1+3=4

- (f) Let  $f$  be a positive, decreasing function on  $\{t : t \geq 1\}$ . Show that the series

$$\sum_{k=1}^{\infty} f(k)$$



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converges if and only if the improper integral

$$\int_1^{\infty} f(t)dt = \lim_{b \rightarrow \infty} \int_1^b f(t)dt$$

5

exists.

Or

Show that the series

$$\sum_{n=1}^{\infty} \cos n$$

is divergent.

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