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1 SEM TDC MTMH (CBCS) C 2

2022

(Nov/Dec)

MATHEMATICS

(Core)

Paper : C–2

(Algebra)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* State the modulus of the complex number $(1+\cos\theta+i\sin\theta)^5$. 1

(b) If $\cos\alpha+\cos\beta+\cos\gamma=0$

$$=\sin\alpha+\sin\beta+\sin\gamma$$

then show that

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$$ 2

(c) Show that

$$(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1} \cos\frac{n\pi}{4}$$ 3

Or

If $\text{cis}\theta = \cos\theta + i\sin\theta$ and $x = \text{cis}\alpha$; $y = \text{cis}\beta$; $z = \text{cis}\gamma$ with $x + y + z = 0$, show that $x^{-1} + y^{-1} + z^{-1} = 0$.

(d) Show that the product of n-numbers of nth root of unity is $(-1)^{n-1}$ and their sum is zero. 4

2. (a) Explain why the set of integers with the relation 'less than or equal to' (≤) is not an equivalence relation. 1

(b) Give an example of a bijective map. 1

(c) Given $f(x) = |x|$, show that $(f \circ f)(x) = f(x)$. 2

(d) If g.c.d $(a, b) = d$, show that

$$\text{g.c.d.}\left(\frac{a}{d}, \frac{b}{d}\right) = 1$$ 2

(e) Show that the relation of equality on the set of integers is an equivalence relation. 3

(f) Use mathematical induction to show that (any *one*)—

(i) 2 is a factor of $5^n - 3^n \; \forall n \in \mathbb{N}$;

(ii) $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n}{2}(n+1)\right]^2$. 3

(g) Show that if $f : X \to Y$ is a bijection, then $\exists$ a map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are identity maps. 3

(h) Let $k > 0$ be an integer and j be any integer. Then show that $\exists$ unique integers q and r such that $j = kq + r$ where $0 \leq r < k$. 5

(i) Show that if a is an odd integer, then $a^{2^n} \equiv 1 (\text{mod}\, 2^{n+2})$ for any $n \geq 1$. 5

3. *(a)* State whether true or false : 1

Each matrix is row equivalent to one and only one reduced Echelon matrix.

(b) Fill in the blank : 1

The equation $x = \alpha u + \beta v$ where α and β are fixed scalars and neither u nor v is a multiple of the other, geometrically represents _____ through the origin.

(c) Solve

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and state the nature of the two non-zero vectors. 1+1=2

(d) State whether the following vectors are linearly dependent or independent by inspection justifying the region thereof : 1+1=2

$$\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

(e) Show that $\forall u, v, w \in \mathbf{R}^n$,

$$(u + v) + w = u + (v + w).$$ 2

(f) Reduce the matrix

$$\begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$$

to row reduced Echelon form using forward and backward phases of row operations. 4

(g) Solve the following system by reducing the augmented matrix to row reduced Echelon form indicating the basic and free variables : 4

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

(h) For an $m \times n$ matrix A, if $u, v \in \mathbf{R}^n$, and c is any scalar, show that—

(i) $A(u+v) = Au + Av$;

(ii) $A(cu) = c(Au)$. 2+2=4

4. (a) For a linear transformation T, show that $T(0) = 0$. 1

(b) For the linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^2$ given by $T(x) = Ax$, state the order of the matrix A. 1

(c) For $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, give the geometric description of the transformation $x \mapsto Ax$. 2

(d) Show that the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx$ is a linear transformation. 2

(e) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Show that $\exists$ a unique matrix A such that $T(x) = Ax$ $\forall x \in \mathbb{R}^n$. 3

(f) If A is an invertible $n \times n$ matrix, then $\forall\ b \in \mathbb{R}^n$, show that the matrix equation $Ax = b$ has the unique solution $x = A^{-1}b$. 3

(g) Show that null A is a subspace of $\mathbb{R}^n$. 4

(h) Find the eigenvalues of

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$ 4

(i) Find the bases for col A and null A stating their dimensions where 5

$$A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}$$

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