5 SEM TDC DSE MTH (CBCS)
2.1/2.2/2.3/2.4 (H)

2024

(November)

MATHEMATICS

(Discipline Specific Elective)

(For Honours)

Paper : DSE–2.1/2.2/2.3/2.4

The figures in the margin indicate full marks for the questions

Paper : DSE–2.1

(MATHEMATICAL MODELLING)

Full Marks : 60

Pass Marks : 24

Time : 3 hours

1. (a) Find the value of $\Gamma\left(\frac{5}{2}\right)$. 1

(b) Is the point $x=0$ an ordinary point of the equation

$$y'' + x^2 y' + x^{1/2} y = 0\,?$$ 1

(c) Find the value of $L\{\cosh kx\}$. 2

(d) Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s-a)}$$ 3

2. (a) Investigate the nature of the point $t=0$ for the differential equation

$$t^4 y'' + (1-\cos t)y + (t^2 \sin t)y' = 0$$ 2

(b) Find a power series solution of the differential equation $(x-3)y' + 2y = 0$. Determine the radius of convergence of the resulting series. 4+1=5

3. (a) Use Laplace transforms to solve the initial value problem

$$x'' - x' - 6x = 0;\ x(0) = 2,\ x'(0) = -1$$ 5

Or

$$x'' + 8x' + 15x = 0;\ x(0) = 2,\ x'(0) = -3$$

(b) Find the Frobenius series solutions of

$$xy'' + 2y' + xy = 0$$ 6

4. *(a)* Define 'goal programs'. 1

(b) Use the linear congruence method to generate 10 random numbers using $a = 5$, $b = 1$ and $c = 8$. 2

(c) Why is sensitivity analysis important in linear programming? 3

5. *(a)* Define feasible solution. 1

(b) Solve the following model algebraically : 4

$$\text{Maximize } 25x + 30y$$

subject to

$$20x + 30y \leq 690$$
$$5x + 4y \leq 120$$
$$x, y \geq 0$$

(c) Explain middle-square method and use it to generate random numbers taking $x_0 = 2041$. Does this method have any drawbacks? Illustrate. 2+3+1=6

6. Answer any *three* of the following questions : 6×3=18

(a) Using Monte Carlo simulation, write an algorithm to calculate the volume of the sphere $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant, $x > 0$, $y > 0$, $z > 0$.

(b) Use the golden-section search method with a tolerance of $t = 0{\cdot}2$ to minimize

$$f(x) = x^2 + 2x, \; -3 \le x \le 6$$

(c) Use the curve-fitting criterion to minimize the sum of the absolute deviations for the model $y = ax^3$ and data set

x	:	7	14	21	28	35	42
y	:	8	41	133	250	280	297

(d) Using simplex method, solve the following problem :

Maximize $3x + y$

subject to

$$2x + y \le 6$$
$$x + 3y \le 9$$
$$x, y \ge 0$$

(e) Fit the model $y = cx$ to the following data using Chebyshev's criterion to minimize the largest deviation :

x	:	1	2	3
y	:	2	5	8

Paper : DSE–2.2

(MECHANICS)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

UNIT—I

1. (a) Define couple moment. A force

$$\vec{F} = (10\hat{i} + 6\hat{j} + 3\hat{k})$$

acts at position (3, 0, 2). At point (0, 2, –3), an equal but opposite force $-\vec{F}$ acts. Calculate the couple moment. 1+2=3

(b) Show that any number of coplanar couples acting on a body is equivalent to a single couple whose moment is equal to the algebraic sum of moments of the couples. 6

Or

Derive the general equations of equilibrium.

(c) An electric light fixture weighing 15 N hangs from a point *C* by two strings *AC* and *BC*. The string *AC* is inclined at 60° to the horizontal and *BC* at 45° to the horizontal. Draw the free-body diagram and determine forces in the strings *AC* and *BC*. 5

(d) A parallel system of forces is such that a 20-N force acts at position $x = 10$ m, $y = 3$ m; a 30-N force acts at position $x = 5$ m, $y = -3$ m; a 50-N force acts at position $x = -2$ m, $y = 5$ m.

(i) If all forces act along negative z-direction, give the simplest resultant force and its line of action.

(ii) If the 50-N force acts along positive z-direction and the others act along negative direction, find the resultant. 4+2=6

UNIT—II

2. (a) What do you mean by coefficient of friction? Write down the dimension of coefficient of friction. 2+2=4

(b) The horizontal position of the 500 kg rectangular block of concrete is adjusted by the 5° wedge under the action of the force *P*. If the coefficient of friction for both wedge surfaces is 0·30 and the coefficient of friction between horizontal surface and block is 0·60, then determine the least force *P* required to move the block. 6

Or

A strongbox of mass 75 kg rests on a floor. The static coefficient of friction for the contact surface is 0·20. Calculate the largest force P and highest position h for applying this force that will not allow the strongbox to either slip on the floor or to tip.

(c) Find the centroid of the area under the half-sine wave. 5

(d) Show that

$$I_{xx} = \frac{bh^3}{12}, I_{yy} = \frac{b^3h}{12} \text{ and } I_{xy} = \frac{b^2h^2}{24}$$

for the right-angled triangle of base b and height h. 5

(e) State and prove the theorem of Pappus-Guldinus. 5

UNIT—III

3. (a) Write down the statement of law of conservation of mechanical energy. 2

(b) Define constant force field. 1

(c) Given the following conservative force field

$$\vec{F} = (10z + y)\hat{i} + (15yz + x)\hat{j} + \left(10x + \frac{15y^2}{2}\right)\hat{k}$$

Find the force potential. Also, calculate the work done by $\vec{F}$ on a particle going from

$$\vec{r_1} = 10\hat{i} + 2\hat{j} + 3\hat{k} \text{ to } \vec{r_2} = -2\hat{i} + 4\hat{j} - 3\hat{k}$$

4+3=7

Or

Show that the moment of the resultant force on a particle about a point, fixed in an inertial reference, is equal to the time rate of change of moment of the linear momentum of the particle relative to the inertial reference frame. 7

4. (a) State and prove Chasles' theorem. 7

(b) A cylinder of radius R rotates about its own axis with an angular speed ω. If the total mass is M, show that the kinetic energy is

$$\frac{1}{4}MR^2\omega^2$$

6

(c) Establish the relation between acceleration vectors of a particle for two systems of references moving arbitrarily relative to each other. 6

(d) Derive the moment of momentum equation for a single particle. 6

Paper : DSE–2.3

(NUMBER THEORY)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

1. (a) Exhibit a complete residue system modulo 7 consisting of multiples of 3. 2

(b) If $\tau(n) = 2$, then what conclusion can be drawn for the number n? 1

(c) Write the values of $\mu(49)$ and $\left[-\frac{1}{3}\right]$. 2

(d) How many primitive roots of 9 are there? 1

(e) Find the value of the Legendre symbol

$$\left(\frac{2}{17}\right)$$ 1

(f) What do you understand by primitive root of an integer n? 1

2. (a) Find the solutions in positive integers for any *one* of the following : 5

(i) $18x + 5y = 48$

(ii) $15x + 7y = 111$

(b) Prove that $a^{21} \equiv a \pmod{15}$ for all a. 3

3. Answer any *two* of the following : 5×2=10

(a) Arrange the integers $2, 3, 4, \cdots, 21$ in pairs a and b that satisfy

$$ab \equiv 1 \pmod{23}$$

(b) Solve the following simultaneous congruences :

$$x \equiv 5 \pmod{7}$$
$$x \equiv 7 \pmod{11}$$
$$x \equiv 3 \pmod{13}$$

(c) State and prove Wilson's theorem. Is the converse of Wilson's theorem true?

4. (a) What is meant by number theoretic function? When is it said to be a multiplicative function? 1+1=2

(b) Find the value of $\tau(900)$. 3

(c) Prove that the Mobius μ function is a multiplicative function. 3

(d) Prove that for a prime p

$$\phi(p^k) = p^k\left(1-\frac{1}{p}\right).$$

Hence find $\phi(32)$. 4

5. Answer any *three* of the following : 5×3=15

(a) For each positive integer $n \geq 1$, prove that

$$n = \sum_{d|n} \phi(d)$$

(b) State and prove Mobius inversion formula.

(c) For $n > 1$, prove that the sum of the positive integers less than n and relatively prime to n is

$$\frac{1}{2}n\phi(n)$$

(d) If $\gcd(m, n) = 1$, then prove that the set of positive divisors of mn consists of all products $d_1 d_2$ where $d_1 | m$, $d_2 | n$ with $\gcd(d_1, d_2) = 1$.

6. (a) Write the conditions for which an integer n fails to have primitive root. 2

(b) Define the words 'plaintext' and 'ciphertext'. 2

(c) Determine whether 2 is quadratic residue or non-residue of 13. 3

(d) Find a Pythagorean triple of the form 16, y, z. 2

(e) Let p be an odd prime and $(a, p) = 1$. Prove that

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$$ 3

7. Answer any *three* of the following : $5 \times 3 = 15$

(a) Solve :

$$x^2 + 7x + 10 \equiv 0 \pmod{11}$$

(b) If order of a modulo $p = 3$, where p is prime, then show that order of $(a+1)$ modulo p is 6.

(c) If p is an odd prime, then there are precisely $\frac{(p-1)}{2}$ quadratic residues and $\frac{(p-1)}{2}$ quadratic non-residues of p.

Prove it.

(d) Solve :

$$x^2 \equiv 7 \pmod{3^3}$$

(e) Prove that there are infinitely many primes of the form $8k - 1$.

Paper : DSE–2.4

(BIOMATHEMATICS)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

UNIT—I

1. Answer any *two* of the following questions : 7½×2=15

(a) A population is originally 100 individuals, but because of the combined effects of births and deaths, it triples each hour.

(i) Make a table of population size for $t=0$ to 5, where t is measured in hours.

(ii) Give two equations modeling the population growth by first expressing P_{t+1} in terms P_t and then expressing ΔP in terms of P_t.

(iii) What can you say about the birth and death rates for this population?

(b) In the early stages of the development of a frog embryo, cell division occurs at a fairly regular rate. Suppose you observe that all cells divide, and hence the number of cells double, roughly every half-hour.

(i) Write down an equation modeling this situation. You should specify how much real-world time is required by an increment of 1 in t and what the initial number of cells is.

(ii) Produce a table and graph of the number of cells as a function of t.

(c) Obtain a simple prey-predator model explaining in detail the assumptions taken. Also find the equilibrium positions.

UNIT—II

2. Answer any *two* of the following questions : $7\frac{1}{2}\times2=15$

(a) Consider the SI epidemic model. If the contact rate is 0·001 and the number of susceptibles is 2000 initially, determine—

(i) the number of susceptibles left after 3 weeks;

(ii) the density of susceptible when the rate of appearance of new cases is a maximum;

(iii) the time (in week) at which the rate of appearance of new cases is a maximum;

(iv) the maximum rate of appearance of new cases.

(b) In an SIS model, if the infection is spread only by a constant number of carriers, then show that

$$I(t) = \left(I_0 - \frac{\alpha CN}{\alpha C + \beta}\right) e^{[-(\alpha C + \beta)t]} + \frac{\alpha CN}{\alpha C + \beta}$$

where I and C are the number of infectives and carriers; N is total population; α and β are contact rate and susceptible rate respectively; I_0 is the infectives at $t = 0$.

(c) Let x and y respectively denote the proportion of susceptibles and carriers in a population. Suppose the carriers are indentified and removed from the population at a rate β, so that

$$\frac{dy}{dt} = \beta y$$

Suppose also that the disease spreads at a rate proportional to the product of x and y, thus

$$\frac{dx}{dt} = -\alpha xy$$

(i) Determine the proportions of carriers at any time t, where $y(0) = y_0$.

(ii) Use (i) above to find the susceptibles at time t, where $x(0) = x_0$.

(iii) Find the proportion of population that escapes the epidemic.

UNIT—III

3. Answer any *two* of the following questions : $7\frac{1}{2} \times 2 = 15$

(a) Consider the competition models for two species with populations N_1 and N_2

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - b_{21}\frac{N_1}{K_2}\right)$$

where only one species N_1 has limited carrying capacity. Investigate their stability and sketch the phase-plane trajectories. [Here, K_1, K_2 are carrying capacities; r_1, r_2 are linear birth rates of the populations N_1 and N_2 respectively; b_{12}, b_{21} measure the competitive effects of N_2 on N_1 and N_1 on N_2 respectively.] $4+3\frac{1}{2}=7\frac{1}{2}$

(b) What is Routh-Hurwitz criterion? Explain with reference to multiple species communities. $2+5\frac{1}{2}=7\frac{1}{2}$

(c) Discuss bifurcation and limit cycle with respect to any biological model. $7\frac{1}{2}$

UNIT—IV

4. Answer any *two* of the following questions : $7\frac{1}{2}\times 2=15$

(a) Write a short note on any *one* of the following :

(i) One-species model with diffusion

(ii) Two-species model with diffusion

(b) For a blood vessel of constant radius R, length L and driving force $P=p_1-p_2$, show that the average velocity of the flow is equal to half of the maximum velocity and the resistance is proportional to $\frac{L}{R^4}$.

(c) Consider arterial blood viscosity $\mu=0\cdot027$ poise.

If the length of the artery is 2 cm, radius 8×10^{-3} cm and $P=p_1-p_2=4\times 10^3$ dynes/cm^2, then find—

(i) $q_z(r)$ and the maximum peak velocity of blood;

(ii) the shear stress at the wall.

(Here q_z denotes velocity along z-axis, p_1 and p_2 denote pressure at two ends of the artery.)

UNIT—V

5. Answer any *two* of the following questions : 10×2=20

(a) Let D & d and W & w respectively denote allele for tall & dwarf and round & wrinkled seeds of peas. Find the outcome of the product $DdWw \times ddWw$ using Punneet square or using probability. Also find the probability that the progeny of $DdWw \times ddWw$ is dwarf with round seeds. 6+4=10

(b) Explain in detail the Hardy-Weinberg equilibrium, mentioning the assumptions considered for the equilibrium.

(c) Compare and contrast stage structure model with age structure model.

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