Total No. of Printed Pages—16

5 SEM TDC STSH (CBCS) C 11 (N/O)

2024

(November)

STATISTICS

(Core)

Paper : C–11

(Stochastic Processes and Queuing Theory)

The figures in the margin indicate full marks for the questions

(New Course)

Full Marks : 55

Pass Marks : 22

Time : 3 hours

1. Choose the correct answer from the following alternatives : 1×7=7

(a) Consider a Markov chain $\{X_n, n \geq 0\}$ with discrete state space. If the transition probabilities are independent of n, then the Markov chain is said to be

(i) independent

(ii) homogeneous

(iii) reducible

(iv) non-homogeneous

(b) If two states of a Markov chain are accessible from each other, then they are

(i) communicating states

(ii) transient states

(iii) absorbing states

(iv) periodic states

(c) Following is the t.p.m. for a Markov chain :

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1-2a & 2a & 0 \\ a & 1-2a & a \\ 0 & 2a & 1-2a \end{bmatrix} \end{array}$$

Then the

(i) value of a is any real number

(ii) value of a is any positive real number

(iii) value of a is less than 0·5

(iv) value of a is in [0, 0·5]

(d) If $\{N(t)\}$ is a Poisson process, then the correlation coefficient between $N(t)$ and $N(t+s)$ is

(i) $\{t(t+s)\}^{3/2}$

(ii) $\dfrac{t}{t+s}$

(iii) $\left\{\dfrac{(t+s)}{t}\right\}^{1/2}$

(iv) $\left\{\dfrac{t}{(t+s)}\right\}^{1/2}$

(e) If $N_1(t)$ and $N_2(t)$ are two independent Poisson processes with parameters λ_1 and λ_2 respectively, then $N_1(t) - N_2(t)$ is

(i) a Poisson process with rate $\lambda_1 + \lambda_2$

(ii) a Poisson process with rate $\lambda_1 - \lambda_2$

(iii) a Poisson process with rate $\dfrac{\lambda_1}{\lambda_2}$

(iv) not a Poisson process

(f) $M/M/1$: model follows

(i) geometric distribution

(ii) exponential distribution

(iii) Poisson distribution

(iv) negative exponential distribution

(g) The term 'jockeying' in queuing theory refers to

(i) not entering the long queue

(ii) leaving the queue

(iii) shifting from one queue to another parallel queue

(iv) None of the above

2. Answer the following questions in brief : $2 \times 6 = 12$

(a) Define a stochastic process with an example.

(b) State Chapman-Kolmogorov theorem.

(c) Define irreducible Markov chain and closed set.

(d) Define transient and persistent states.

(e) Prove that the interval between two successive occurrences of a Poisson process $\{N(t), t \geq 0\}$ having parameter λ has a negative exponential distribution with mean $\frac{1}{\lambda}$.

(f) What do the letters in the symbolic representation $(a/b/c):d/e$ of a queuing model represent?

3. (a) (i) Define bivariate probability generating function. Given the bivariate p.g.f. of (X, Y) as

$$P(s_1, s_2) = \exp[-a - b - c + as_1 + bs_2 + cs_1s_2]$$

Find the p.g.f. of $X + Y$ and identify the distribution. 1+1=2

(ii) The r.v. X has logarithmic series distribution if

$$p_k = P(X = k) = \frac{Xq^k}{k}; \ k = 1, 2, 3, \ldots$$

$$X = -\frac{1}{\log p}; \quad 0 < q = 1 - p < 1$$

Find the p.g.f. of X. Also, find mean and variance. 3

Or

(b) If $\{N(t)\}$ is a Poisson process and $s < t$, then prove that

$$P\{N(s) = k / N(t) = n\} = \binom{n}{k}\left(\frac{s}{t}\right)^k \left\{1 - \left(\frac{s}{t}\right)\right\}^{n-k}$$ 5

4. Answer any *four* questions from the following : 4×4=16

(a) Interpret the Gambler's ruin problem as a Markov chain.

(b) Consider the Markov chain with three states $S = \{1, 2, 3\}$, that has the following transition matrix :

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(i) Draw the transition diagram of the Markov chain.

(ii) If $P(X_1 = 1) = P(X_2 = 2) = \frac{1}{4}$, then find $P(X_1 = 3, X_2 = 2, X_3 = 1)$. 2+2=4

(c) Let $\{X_n, n \geq 0\}$ is a Markov chain with t.p.m.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

and initial distribution $P(X_0 = j) = \frac{1}{2}$; $j = 0, 1$

Compute—

(i) $P(X_1 = 1)$;

(ii) $P(X_1 = 0)$;

(iii) $P(X_2 = 1)$;

(iv) $P(X_2 = 0)$. 1+1+1+1=4

(d) Define ergodic state in Markov chain. Let $\{X_n, n \geq 0\}$ be a Markov chain having state space $s = \{1, 2, 3, 4\}$ and t.p.m.

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Show that state '1' is ergodic. 1+3=4

(e) Define periodicity of a Markov chain. Show that the states of the Markov chain with t.p.m.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

are periodic and persistent non-null.

1+3=4

5. (a) (i) Show that the sum of two independent Poisson processes is a Poisson process. 2

(ii) State the differential-difference equation of linear growth process and hence using the equation, show that $M'(t) = (\lambda - \mu) M(t)$, where $M(t) = \sum_{n=1}^{\infty} n \cdot P_n(t)$, λ and μ are respectively the birth and death rates of the process $\{X_t\}$. 5

Or

(b) Show that if the intervals between successive occurrences of an event E are independently distributed with a common exponential distribution with mean $\frac{1}{\lambda}$, then the event E forms a Poisson process with mean λt. 7

6. *(a)* Define a queuing system. Discuss the basic features which characterize a queuing system.
"Queue is a management of congestions." Elucidate the statement. 4+4=8

Or

(b) In case of $(M / M / 1) : (N / FCFS)$ queuing model, derive the steady-state probability distribution and obtain the expression for—
(i) expected number of customers in the system;
(ii) expected number of customers in the queue. 6+2=8

(Old Course)

Full Marks : 50

Pass Marks : 20

Time : 2 hours

1. Choose the correct answer from the following alternatives : 1×6=6

(a) Consider a Markov chain $\{X_n, n \geq 0\}$ with discrete state space. If the transition probabilities are independent of n, then the Markov chain is said to be

(i) independent

(ii) homogeneous

(iii) reducible

(iv) non-homogeneous

(b) If two states of a Markov chain are accessible from each other, then they are

(i) communicating states

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(iii) absorbing states

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(c) If $\{N(t)\}$ is a Poisson process, then the correlation coefficient between $N(t)$ and $N(t+s)$ is

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(d) If $N_1(t)$ and $N_2(t)$ are two independent Poisson processes with parameters λ_1 and λ_2 respectively, then $N_1(t)-N_2(t)$ is

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(ii) a Poisson process with rate $\lambda_1-\lambda_2$

(iii) a Poisson process with rate $\frac{\lambda_1}{\lambda_2}$

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(e) $M/M/1$: model follows

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(f) The term 'jockeying' in queuing theory refers to

(i) not entering the long queue

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2. Answer the following questions in brief : 2×5=10

(a) Define a stochastic process with an example.

(b) State Chapman-Kolmogorov theorem.

(c) Define irreducible Markov chain and closed set.

(d) Define transient and persistent states.

(e) What do the letters in the symbolic representation $(a/b/c):d/e$ of a queuing model represent?

3. (a) (i) Define bivariate probability generating function. Given the bivariate p.g.f. of (X, Y) as

$$P(s_1, s_2) = \exp[-a - b - c + as_1 + bs_2 + cs_1 s_2]$$

Find the p.g.f. of $X + Y$ and identify the distribution. 1+1=2

(ii) The r.v. X has logarithmic series distribution if

$$p_k = P(X = k) = \frac{Xq^k}{k}; \; k = 1, 2, 3, \ldots$$

$$X = -\frac{1}{\log p}; \quad 0 < q = 1 - p < 1$$

Find the p.g.f. of X. Also, find mean and variance. 3

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Compute—

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(iv) $P(X_2 = 0)$. 1+1+1+1=4

(d) Define ergodic state in Markov chain. Let $\{X_n, n \geq 0\}$ be a Markov chain having state space $S = \{1, 2, 3, 4\}$ and t.p.m.

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

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(e) Define periodicity of a Markov chain. Show that the states of the Markov chain with t.p.m.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

are periodic and persistent non-null. 1+3=4

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Or

(b) Show that if the intervals between successive occurrences of an event E are independently distributed with a common exponential distribution with mean $\frac{1}{\lambda}$, then the event E forms a Poisson process with mean λt. 6

6. *(a)* Define a queuing system. Discuss the basic features which characterize a queuing system.

"Queue is a management of congestions." Elucidate the statement.

4+3=7