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**6 SEM TDC MTMH (CBCS) C 14**

**2025**

**( May )**

**MATHEMATICS**

**( Core )**

**Paper : C-14**

**( Ring Theory and Linear Algebra—II )**

Full Marks : 80

Pass Marks : 32

**Time : 3 hours**

*The figures in the margin indicate full marks  
for the questions*

1. (a) If  $F$  is commutative, then write the condition such that  $F[x]$  is invertible. 1
- (b) Prove that every Euclidean domain possesses unity. 2
- (c) Show that  $x^2 + 3x + 2$  has four zeros in  $Z_6$ . 2

- (d) Let  $F$  be a field. Then prove that the ring of polynomial  $F[x]$  is principal ideal domain (PID). 4

- (e) Prove that a polynomial of degree  $n$  over a field has at most  $n$  zeros, counting multiplicity. 6

Or

Let  $F$  be a field and let  $f(x), g(x) \in F[x]$  with  $g(x) \neq 0$ . Then prove that there exist unique polynomials  $q(x)$  and  $r(x)$  in  $F[x]$  such that  $f(x) = g(x)q(x) + r(x)$  and either  $r(x) = 0$  or  $\deg r(x) < \deg g(x)$ .

2. (a) What is the inverse of  $1 + \sqrt{2}$  in  $Z[\sqrt{2}]$ ? 1
- (b) Define Euclidean domain. 1
- (c) Test the irreducibility of the polynomial  $x^5 + 9x^4 + 12x^2 + 6$  in  $\mathbb{Q}$ . 2
- (d) Prove that in a principal ideal domain, an element is irreducible if and only if it is a prime. 5
- (e) Define unique factorization domain and prove that every field is unique factorization domain. 1+5=6



Or

Prove that  $Z[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in Z\}$  is a Euclidean domain.

6

3. (a) Write when two linear functionals are said to be equal on a vector space  $V(F)$ .

1

(b) Define invariant subspace.

1

(c) If  $S_1$  and  $S_2$  are two subsets of a vector space  $V(F)$  such that  $S_1 \subseteq S_2$ , then show that  $S_2^\circ \subseteq S_1^\circ$ .

2

(d) Prove that the subspace spanned by two subspaces each of which is invariant under some linear operator  $T$ , is itself invariant under  $T$ .

3

(e) Let  $V$  be an  $n$ -dimensional vector space over the field  $F$  and let

$$\beta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

be a basis for  $V$ . Then prove that there is a uniquely determined basis

$$\beta' = \{f_1, f_2, \dots, f_n\}$$

for  $V'$  such that  $f_i(\alpha_j) = \delta_{ij}$ .

6

Or

Let  $V$  be finite dimensional vector space over the field  $F$  and let  $W$  be a subspace of  $V$ . Then prove that

$$\dim W + \dim W^\circ = \dim V$$

4. (a) Write about the eigenvalues and eigenvectors of the identity matrix. 1
- (b) If  $V$  is  $n$ -dimensional vector space, then what is the condition that the linear operator  $T$  is diagonalizable? 1
- (c) Test the diagonalizability of the following matrix : 4

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

- (d) Define minimal polynomial and show that the minimal polynomial of the real matrix

$$\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

is  $(x-1)(x-2)$ .

1+5=6



Or

If  $f(x)$  be the characteristic polynomial of  $T$ , then prove that  $f(T) = \hat{0}$ . 6

5. (a) Write the only vector that is orthogonal to itself. 1

(b) Define orthogonal complement. 1

(c) If  $\alpha, \beta$  are vectors in an inner product space  $V$ , then prove that

$$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\| \quad 4$$

Or

If  $W_1$  and  $W_2$  are subspaces of a finite dimensional inner product space, then prove that

$$(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$$

(d) If  $\beta = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$  is any finite orthonormal set in an inner product space  $V$ , and if  $\beta$  is any vector in  $V$ , then prove that

$$\sum_{i=1}^m |(\beta, \alpha_i)|^2 \leq \|\beta\|^2 \quad 6$$

Or

In an inner product space, prove that

$$|(\alpha, \beta)| \leq \|\alpha\| \|\beta\|$$

6. (a) Write the two self-adjoint operators on any inner product space  $V(F)$ . 1
- (b) Define normal operator. 1
- (c) If  $T_1$  and  $T_2$  are normal operators on an inner product space with the property that either commutes with the adjoint of the other, then prove that  $T_1 T_2$  is also normal operator. 2
- (d) Let  $V$  be the direct sum of its subspaces  $W_1$  and  $W_2$ . If  $E_1$  is the projection on  $W_1$  along  $W_2$ , and  $E_2$  is the projection on  $W_2$  along  $W_1$ , then prove that—
- (i)  $E_1 + E_2 = I$  ;
- (ii)  $E_1 E_2 = \hat{O}$ ,  $E_2 E_1 = \hat{O}$ . 4
- (e) If  $T_1$  and  $T_2$  are self-adjoint linear operators on an inner product space  $V$ , then prove that (i)  $T_1 + T_2$  is self-adjoint and (ii) if  $T_1 \neq \hat{O}$  and  $a$  is a non-zero scalar, then  $aT_1$  is self-adjoint iff  $a$  is real. 5



( 7 )

*Or*

Apply the Gram-Schmidt process to the vectors  $(1, 0, 0)$ ,  $(1, 1, 0)$ ,  $(1, 1, 1)$  to obtain an orthonormal basis for  $V_3(R)$  with the standard inner product.

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