5 SEM TDC MTH M 2

2019

(November)

MATHEMATICS

(Major)

Course : 502

(Linear Algebra and Number Theory)

Full Marks : 80

Pass Marks : 32/24

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. *(a)* Write the necessary and sufficient condition for existence of non-trivial solution of the homogeneous system $Ax = 0$. 1

(b) Let F be a field and V be the set of all ordered pairs (x, y), $x \in F$, $y \in F$. Define—

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$;

(ii) $\alpha(x, y) = (\alpha x, y)$, where $\alpha \in F$.

Show that V cannot be a vector space with respect to the operations defined as above. 2

(c) Show that the set

$$S = \{1, x, x^2, \cdots, x^n\}$$

of n polynomials is a basis set for the vector space $P_n(\mathbb{R})$ of all polynomials of degree $\leq n$ with real coefficients. 3

(d) Determine the consistency of the following system of linear equations : 3

$$3x_1 + 4x_2 - x_3 + 2x_4 = 1$$
$$x_1 - 2x_2 + 3x_3 + x_4 = 2$$
$$3x_1 + 14x_2 - 11x_3 + x_4 = 2$$

(e) Find the general solution of the following system of linear equations : 5

$$2x_1 + 3x_2 - x_3 + 2x_4 = 3$$
$$x_1 + 2x_2 + x_3 - 3x_4 = 1$$
$$2x_1 + x_2 - 6x_3 + x_4 = -2$$

(f) Let V be the vector space of all 2×2 matrices over a field F. Let

$$W_1 = \left\{ \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \middle| \; x, y, z \in F \right\} \text{ and}$$

$$W_2 = \left\{ \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \middle| \; x, z \in F \right\}$$

Show that W_1 and W_2 are subspaces of V. Also compute $W_1 + W_2$ and $W_1 \cap W_2$. 2+2+2=6

2. (a) Define affine space of a vector space. Is it a vector subspace of V? 1+1=2

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(x, y) = (x, y, xy), \; x, y \in \mathbb{R}$$

Prove that T cannot be a linear transformation. 2

(c) What do you understand by nullity of a linear transformation? Find the nullity of T, where $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2) = (x_1 + x_2, \; x_1 - x_2, \; x_2)$$

given T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. 1+2=3

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation defined by

$$T(x, y) = (x, y, x+y, x-y), \; x, y \in \mathbb{R}$$

Find the matrix of T with respect to the standard bases. 3

(e) Let V and W be finite dimensional vector spaces and $T : V \to W$ be a linear transformation. Then prove that

$$\dim V = \text{rank } T + \text{nullity } T$$ 5

(f) Let W be a vector subspace of a vector space V over $\mathbb{R}$. Let $v_1 + W$ and $v_2 + W$ be cosets of W in V. Then prove that

either $(v_1 + W) \cap (v_2 + W) = \phi$

or $v_1 + W = v_2 + W$

Moreover, $v_1 + W = v_2 + W$ iff $v_1 - v_2 \in W$. 5

Group—B

(Number Theory)

(*Marks* : 40)

3. (*a*) What is well ordering principle of positive integers? 1

(*b*) Answer any *two* from the following questions : 3×2=6

(*i*) Prove that the square of any odd integer is of the form $8k+1$.

(*ii*) If $(a, b)=d$, then show that $\frac{a}{d}$ and $\frac{b}{d}$ are relatively prime.

(*iii*) Suppose a, b, x are integers such that $a \mid bx$. If $(a, b)=1$, then prove that $a \mid x$.

4. (*a*) Answer any *two* from the following questions : 3×2=6

(*i*) List 15 consecutive natural numbers so that none of them is a prime number.

(*ii*) Evaluate $(a+b,\ p^4)$ given that $(a,\ p^2)=p$ and $(b,\ p^3)=p^2$, where p is a prime.

(*iii*) Find the highest power of 7 dividing 2000!.

(b) Let $p > 1$ and p has the property that if for any $a, b \in Z$, $p \mid ab \Rightarrow p \mid a$ or $p \mid b$. Then prove that p is a prime. 2

5. (a) State Fermat's theorem. Is the converse true? 1+1=2

(b) Answer any *two* from the following questions : 4×2=8

(i) For $a, b \in Z$, prove that $a \equiv b \pmod{n}$ if and only if a and b leave the same remainder upon division by n.

(ii) Using congruence, prove that 41 divides $2^{20} - 1$.

(iii) Using Chinese remainder theorem, solve

$$x \equiv 2 \pmod{3}$$

$$x \equiv 3 \pmod{5}$$

$$x \equiv 2 \pmod{7}$$

(c) Prove that $am \equiv bm \pmod{n}$ if and only if

$$a \equiv b \left(\bmod \frac{n}{(m, n)} \right)$$

Write the necessary condition so that $am \equiv bm \pmod{n}$ gives $a \equiv b \pmod{n}$. 4+1=5

6. (a) What is the domain of definition of an arithmetic function? 1

(b) Prove that $\phi(n) = \frac{n}{2}$, if n is a power of 2. 2

(c) Find the value of $\sigma(180)$. 2

(d) Prove that $\sigma_{-k}(n) = n^{-k}\sigma_k(n)$. 2

(e) If m, n are two relatively prime positive integers, then prove that

$$P(mn) = P(m)^{d(n)} \cdot P(n)^{d(m)}$$

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