5 SEM TDC MTH M 2

2018

(November)

MATHEMATICS

(Major)

Course : 502

(Linear Algebra and Number Theory)

Full Marks : 80

Pass Marks : 32/24

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. *(a)* Choose the correct answer from the brackets to fill in the blank : 1

"A set consisting of a single non-zero vector is _____."

(linearly dependent / linearly independent)

(b) For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$, let $x + y = (x_1 + y_1, x_2 + y_2)$ and $\alpha x = \alpha(x_1, x_2) = (\alpha x_1, 0)$.

Is $\mathbb{R}^2$ a vector space with respect to the above operations? Justify your answer. 1+1=2

(c) Let V be a vector space and X be a non-empty set. Let W be the set of functions $f : X \to V$. On W, define addition and scalar multiplication as follows

$$(f + g)(x) = f(x) + g(x), \; f, g \in W, \; x \in X$$

$$(\alpha \cdot f)(x) = \alpha f(x), \; \alpha \in \mathbb{R}, \; x \in X$$

Then show that W is a vector space. 3

(d) Prove that any basis of a finite dimensional vector space is finite. 4

(e) Let W be the subspace of $\mathbb{R}^4$ generated by the vectors $(1, -2, 5, -3)$, $(2, 3, 1, -4)$ and $(3, 8, -3, -5)$. Find a basis of W. 3

(f) Find for what value of k the vector $u = (1, -2, k)$ in $\mathbb{R}^3$ is a linear combination of the vectors $v = (3, 0, -2)$ and $w = (2, -1, -5)$. 3

(g) Let W_1 and W_2 be two subspaces of a finite dimensional vector space, then show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$ 4

2. *(a)* Let V be a finite dimensional vector space, then prove that any two bases of V have the same number of elements. 4

(b) Examine whether the following mappings are linear or not : 2+2=4

(i) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$T(x, y, z) = 2x - 3y + 4z$$

(ii) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (|x|, 0)$$

(c) Let S and T be two subsets of a vector space $V(F)$, then show that

$$S \subseteq L(T) \Rightarrow L(S) \subseteq L(T)$$ 4

(d) Define line of a vector space. Show that any two distinct points determine a unique line. 1+2=3

(e) Prove that any two n-dimensional vector spaces are isomorphic. 5

GROUP—B

(Number Theory)

(*Marks* : 40)

3. For any integers a, b, c, show that $a|b$, $a|c \Rightarrow a|bx+cy \ \forall\, x, y \in \mathbb{Z}$. 1

4. Answer any *two* from the following : 3×2=6

(a) Let a and b any positive integers, then prove that $(a, b)\cdot[a, b] = ab$.

(b) Using Euclidean algorithm, solve the equation $726x + 275y = 11$.

(c) Prove that there is no natural number in between 0 and 1.

5. (a) Show that there are infinitely many primes of the form $4n+3$. 3

(b) Prove that

$$S_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is never an integer. 3

(c) Let p be a prime and a any integer, then show that either $p|a$ or $(a, p) = 1$. 2

6. (a) State Fermat's little theorem. 1

(b) Show that the relation of congruence on integers is an equivalence relation. 3

(c) Show that $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9. 3

(d) If p is prime, then prove that

$$(p-1)! \equiv -1 \pmod{p}$$ 4

(e) Find all solutions in positive integers of

$$5x + 3y = 52$$ 4

Or

Using Chinese Remainder theorem, solve

$$x \equiv 5 \pmod{18}$$
$$x \equiv -1 \pmod{24}$$
$$x \equiv 17 \pmod{33}$$ 4

7. (a) Prove that

$$P(n) = n^{\frac{d(n)}{2}}$$ 3

(b) Prove that

$$\sum_{d|n} \phi(d) = n, \quad \forall n \in \mathbb{N}$$ 4

(c) What are the positive integers x, y that satisfy the expression $\phi(xy) = \phi(x) + \phi(y)$? 3

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