5 SEM TDC MTH M 2

2017

(November)

MATHEMATICS

(Major)

Course : 502

(Linear Algebra and Number Theory)

Full Marks : 80

Pass Marks : 32/24

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. *(a)* When are two systems of linear equations said to be equivalent? 1

(b) Is the vector space $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Give reasons to your answer. 1+1=2

(c) Prove that if two vectors in a vector space are linearly dependent, then one of them is a scalar multiple of the other. 2

(d) Determine whether the following system is consistent or not : 3

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

(e) Show that the set $H = \{(3t, 2+5t) : t \in Z\}$ cannot be a subspace of the vector space $\mathbb{R}^2$. 3

(f) Find the value of h so that the vector w be in the subspace of $\mathbb{R}^3$ spanned by the vectors v_1, v_2, v_3, where

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \text{ and } w = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

4

(g) Show that the set

$$B = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$$

is a basis of the real vector space $\mathbb{R}^3$. Hence find the coordinates of the vector (a, b, c) with respect to the above basis. 5

Or

If W is a subspace of a finite dimensional vector space V over a field F, then prove that

$$\dim \frac{V}{W} = \dim V - \dim W$$ 5

2. *(a)* Define an affine subspace of a vector space with example. 2

(b) Show that $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ belongs to the null space of $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$. 2

(c) Define null space of a linear transformation. Let U and V be two vector spaces over the same field F and T be a linear transformation from U into V. Then prove that the null space of T is a subspace of U. 1+2=3

(d) Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T(x, y) = (4x - 2y, 2x + y)$$

Find the matrix representation of T relative to the basis $\{(1, 1), (-1, 0)\}$. 4

(e) Let V be the vector space of all complex numbers $a + ib$ over the field of reals $\mathbb{R}$ and let T be a mapping from V to $\mathbb{R}^2$ defined as $T(a + ib) = (a, b)$. Show that T is an isomorphism of V into $\mathbb{R}^2$. 4

(f) Find the range and Kernel of $T:\mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+z \\ x+y+2z \\ 2x+y+3z \end{pmatrix}$$ 5

GROUP—B

(Number Theory)

(*Marks* : 40)

3. (a) When are two integers said to be relatively prime? 1

(b) Prove that if $a|b$ and $b \neq 0$, then $|a| \leq |b|$. 2

(c) Show that the square of any odd integer is of the form $8k+1$. 4

4. Answer any *two* of the following : 3×2=6

(a) Prove that there exists no rational algebraic formula which represents prime numbers only.

(b) Prove that the set of prime numbers is infinite.

(c) Show that $[x]+[-x]=0$ or -1 according as x is an integer or fraction.

5. Write the values of $[-\pi]$ and $\left[\frac{1}{9}\right]$. 2

6. *(a)* Write the reduced set of residues mod 40. 1

(b) Prove that if $a \equiv b \pmod{n}$ and $m|n$, then $a \equiv b \pmod{m}$. 2

(c) Find the remainder when 2^{51} is divided by 7. 2

(d) Find the positive integer solutions of the equation $7x + 19y = 213$. Also determine the number of solutions for this equation. 4+1=5

Or

Solve $9x \equiv 21 \pmod{30}$ in integers and also find the total number of incongruent solutions. 5

(e) Solve the following : 5

$$x \equiv 1 \pmod{3}$$
$$x \equiv 2 \pmod{5}$$
$$x \equiv 3 \pmod{7}$$

7. *(a)* Define $\sigma(n)$ and find $\sigma(2)$. 1+2=3

(b) Prove that if p is a prime, then

$$\phi(p)+\sigma(p)=p\cdot d(p)$$ 2

(c) Prove that if $\sigma_{-k}(n)=n^{-k}\sigma_k(n)$. 3

(d) Evaluate : 2

$$P(10) \text{ and } \mu(24)$$

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