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5 SEM TDC MTH M 2

2016

(November)

MATHEMATICS

(Major)

Course : 502

(Linear Algebra and Number Theory)

Full Marks : 80

Pass Marks : 32 (Backlog)/24 (2014 onwards)

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. *(a)* Write which of the following statements is 'true' and which is 'false' : 1×2=2

(i) "The set containing a linearly independent set of vectors is itself linearly independent."

(ii) "Intersection of two subspaces of a vector space V is always a subspace of V."

(b) Examine whether the vector (2, −5, 3) is in the subspace of $\mathbb{R}^3$ spanned by the vectors (1, −3, 2), (2, −4, −1) and (1, −5, 7). 3

(c) Show that the set

$$S = \{(1, 0), (i, 0), (0, 1), (0, i)\}$$

forms a basis for the vector space V of ordered pairs of complex numbers over the field of real numbers $\mathbb{R}$, i.e., $V = \mathbb{C}^2(\mathbb{R})$. 3

(d) Let V be a finite dimensional vector space of dimension n. Then prove that any set of n linearly independent vectors in V forms a basis for V. 3

(e) Let V be any vector space. Prove that the set $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent if and only if one of the v_i's is a linear combination of the other v_j's where $v_k \in V$, $1 \le k \le n$. 5

(f) Define subspace of a vector space. Prove that the set W defined as

$$W = \{(a, b, 0) : a, b \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^3$. 2+2=4

2. *(a)* Let $l(p; d)$ and $l(q; d)$ be two lines passing through p and q respectively having direction d. Show that $l(p; d) = l(q; d)$ if and only if $(q-p)$ is a multiple of d. 3

(b) Let T be a linear transformation from a vector space U to a vector space V over the field F. Prove that the range of T is a subspace of V. 3

(c) Show that a linear map T from a vector space to another is one-one if and only if $\ker T = \{0\}$. 4

(d) Let V be the vector space of all polynomials in x with coefficients in $\mathbb{R}$ of the form

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$

The differentiation operator D is a linear transformation on V. Write the matrix of D relative to the ordered basis

$$B = \{x^0, x^1, x^2, x^3\}$$ 4

(e) Show that the mapping $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T(a, b) = (a+b, a-b, b)$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Find the rank and nullity of T. 2+2+2=6

GROUP—B

(Number Theory)

(*Marks* : 40)

3. When are two integers said to be relatively prime? 1

4. Answer any *two* from the following : 3×2=6

(a) Use division algorithm to establish that the square of any integer is either of the form $3k$ or $3k+1$.

(b) Prove that if $a \mid bc$ with $\gcd(a, b) = 1$, then $a \mid c$.

(c) Use Euclidean algorithm to obtain integers x and y satisfying the following :

$$\gcd(56, 72) = 56x + 72y$$

5. *(a)* Show that if p is a prime and $p|ab$ then either $p|a$ or $p|b$. 3

(b) Prove that given any positive integer n, there exist n consecutive composite integers. 3

(c) Find the highest power of 5 dividing $100!$. 2

6. (a) Write a complete set of residues modulo 7. 1

(b) If $a \equiv b \pmod{n}$ and the integers a, b, n are all divisible by $d > 0$, then prove that 3

$$\frac{a}{d} \equiv \frac{b}{d}\left(\text{mod}\,\frac{n}{d}\right)$$

(c) If a is an odd integer, then prove that 4

$$a^2 \equiv 1 \pmod{8}$$

(d) Solve $18x + 5y = 48$. 4

(e) Solve the following by using Chinese remainder theorem : 3

$$x \equiv 5 \pmod{4}$$
$$x \equiv 3 \pmod{7}$$
$$x \equiv 2 \pmod{9}$$

7. (a) Evaluate

(i) $\sigma(210)$

(ii) $d(63)$

(iii) $\phi(100)$

where the symbols have their usual meanings. 2×3=6

(b) When is an arithmetic function said to be multiplicative? Prove that σ is a multiplicative function. 1+3=4

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