5 SEM TDC MTH M 2

2014

(November)

MATHEMATICS

(Major)

Course : 502

(Linear Algebra and Number Theory)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following questions : 1×4=4

(a) Define subspace of a vector space.

(b) Let V be a vector space over same field K. Show that

$$k(-v) = -kv, \ k \in F, \ v \in V$$

(c) Define kernel of a linear mapping.

(d) Find the matrix of the linear operator $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x, y) = (2x - 3y,\ x + y)$$

with the usual basis.

2. Answer the following questions :

(a) Determine whether $u = (3, 4)$ and $v = (1, -3)$ are linearly dependent. 2

(b) Let V be a vector space over K and $W \subseteq V$. Show that W is a subspace of V iff

(i) $0 \in W$;

(ii) $u, v \in W \Rightarrow \alpha u + \beta v \in W,\ \forall\, \alpha, \beta \in K$. 3

(c) Find, for what value of k, the vector $u = (1, -2, k)$ in $\mathbb{R}^3$ is a linear combination of the vectors $v = (3, 0, -2)$ and $w = (2, -1, -5)$. 3

(d) Let W be the subspace of $\mathbb{R}^4$ generated by the vectors $(1, -2, 5, -3)$, $(2, 3, 1, -4)$, $(3, 8, -3, -5)$. Find a basis of W. 3

(e) Prove that a vector space V is the direct sum of its subspaces U and W, iff

(i) $V = U + W$

(ii) $U \cap W = \{0\}$ 3

(f) Examine whether the following mappings are linear or not : 4

(i) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$T(x, y, z) = 2x - 3y + 4z$$

(ii) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (|x|, 0)$$

3. Answer any *three* of the following questions : 6×3=18

(a) Show that any two bases of a finite-dimensional vector space have same number of elements.

(b) Let $T : V \to U$ be a linear transformation. Show that

$$\dim V = \text{rank of } T + \text{nullity of } T$$

(c) If V is a finite-dimensional vector space and W is a subspace of V, then show that

$$\dim (V / W) = \dim V - \dim W$$

(d) Prove that the row rank and the column rank of an $m \times n$ matrix $A = (a_{ij})$ are equal.

GROUP—B

(Number Theory)

(*Marks* : 40)

4. Answer the following questions : $1\times4=4$

(a) If g.c.d. $(a, b) = d$, then show that

$$\text{g.c.d.}\left(\frac{a}{d}, \frac{b}{d}\right) = 1$$

(b) If $[n]$ denotes the largest integer $\leq n$, then find the value of

$$\left[\frac{50}{3}\right]+\left[\frac{50}{3^2}\right]+\left[-\frac{50}{3^3}\right]$$

(c) State Fermat's little theorem.

(d) What is the value of $\alpha(n)$, if n is prime?

5. Answer the following questions : $2\times3=6$

(a) Prove that every non-empty subset of $\mathbb{N}$ contains a least element.

(b) If p is a prime and $p \mid ab$, then prove that $p \mid a$ or $p \mid b$, where $a, b \in \mathbb{Z}$.

(c) For any prime p, show that $\phi(p) = p-1$.

6. Answer the following questions : $3\times6=18$

(a) Prove that $an \equiv bn \pmod{m}$, if and only if

$$a \equiv b\left(\text{mod}\,\frac{m}{(m, n)}\right)$$

(b) Find the remainder when $2^{73}+14^3$ is divided by 11.

(c) Solve $5x+11y=92$.

(d) Solve the system

$$x \equiv 6 \pmod{17}$$
$$x \equiv 17 \pmod{24}$$
$$x \equiv 13 \pmod{33}$$

(e) Prove that $P(n)=n^{\frac{d(n)}{2}}$.

(f) Prove that there are infinitely many primes of the form $4n+3$.

7. Answer any *three* of the following : $4\times3=12$

(a) State and prove the division algorithm.

(b) Prove that every positive integer (> 1) can be expressed as a product of primes is unique apart from the order of factors.

(c) Prove that

$$\sum_{d|n}\mu(d)=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n>1\end{cases}$$

(d) State and prove Chinese remainder theorem.

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