5 SEM TDC MTH M 2

2013

(November)

MATHEMATICS

(Major)

Course : 502

(Linear Algebra and Number Theory)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following questions : 1×4=4

(a) Under what condition, two systems of linear equations over the same field are said to be equivalent?

(b) Write the standard basis for the vector space $\mathbb{R}^3(\mathbb{R})$.

(c) Define null space of a linear transformation.

(d) Let $T: V \rightarrow W$ be a linear map given by $T(v) = 0, \forall \; v \in V$. What will be the kernel of T?

2. Answer the following questions :

(a) Find the dimension of the quotient space $\mathbb{R}^3 / W$, where W is the subspace of $\mathbb{R}^3$ spanned by $(1, 1, 0)$ and $(1, 0, 0)$. 2

(b) If α_1 and α_2 are the vectors of $V(F)$ and $a, b \in F$, then prove that

$$\{\alpha_1, \alpha_2, a\alpha_1 + b\alpha_2\}$$

is linearly dependent. 3

(c) Show that the subset

$$W = \{(a, b, c) : a + b + c = 0\}$$

of $\mathbb{R}^3(\mathbb{R})$ is a subspace of $\mathbb{R}^3(\mathbb{R})$. 3

(d) Prove that the intersection of any two subspaces of a vector space is also a subspace of the vector space. 3

(e) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by

$$T(x_1, x_2) = (x_1 + x_2, \; x_2, \; x_2)$$

Find the matrix of T w.r.t. the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. 3

(f) If $\mathbb{R}$ be the field of real numbers, then prove that the vectors (a, b) and (c, d) in $\mathbb{R}^2$ are linearly dependent if and only if $ad - bc = 0$. 4

3. Answer any *three* of the following questions : 6×3=18

(a) Define affine space. Let W be a subspace of a vector space V and $v \in V$ be fixed. Prove that $S = \{v + W \mid v \in V\}$ is an affine space.

(b) Let T be a linear transformation from V into W. Then prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

(c) Show that the mapping $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T(a, b) = (a + b, a - b, b)$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. Find the rank, null space and nullity of T.

(d) If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then prove that $W_1 + W_2$ is finite-dimensional and

$$\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2)$$

GROUP—B

(Number Theory)

(*Marks* : 40)

4. Answer the following questions : 1×4=4

(a) Write the well-ordering principle (WOP) of positive integers.

(b) If $a, b \in Z^+$ and there exists $x, y \in Z$ such that $ax + by = 1$, then write the value of (a, b).

(c) Define Euler's function $\phi(n)$.

(d) Write a reduced set of residues mod 10.

5. Answer the following questions : 2×3=6

(a) Show that the difference between any integer and its cube is always divisible by 6.

(b) If g.c.d. $(a, b) = d$, then prove that

$$\text{g.c.d.}\left(\frac{a}{d}, \frac{b}{d}\right) = 1$$

(c) Under which situation, an arithmetic function is said to be a multiplicative function? Is the function $\sigma(n)$ defined as the sum of the divisors of n, multiplicative?

6. Answer the following questions : 3×6=18

(a) Prove that if $a \equiv b \pmod{m}$ and $a \equiv b \pmod{n}$ $(m, n) = 1$, then

$$a = b \pmod{mn}$$

(b) Solve in integers :

$$7x + 5y = 5$$

(c) By the principle of mathematical induction, prove that $3^{2n} - 1$ is divisible by 8, $\forall\ n \in \mathbb{N}$.

(d) Find the highest power of 5 which is contained in 500!.

(e) Is the system of linear congruence given below solvable? Give reasons for your answer :

$$x \equiv 5 \pmod{8}$$
$$x \equiv 9 \pmod{12}$$
$$x \equiv 3 \pmod{18}$$

(f) Find the value of the following :

(i) $\sigma_2(6)$

(ii) $\phi(700)$

7. Answer any *three* of the following : 4×3=12

(a) If $g = (a, b)$, then prove that there exist integers x and y such that $g = ax + by$.

(b) State Fermat's little theorem. Using Fermat's little theorem, find the remainder when $13^{73} + 14^{3}$ is divided by 11.

(c) Prove that there are infinitely many primes.

(d) Prove that for $n \geq 1$, $n \in \mathbb{Z}$

$$n = \sum_{d|n} \phi(d)$$

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