3 SEM TDC MTH M 1

2019

(November)

MATHEMATICS

(Major)

Course : 301

[Analysis—I (Real Analysis)]

Full Marks : 80
Pass Marks : 32/24

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Differential Calculus)

(*Marks* : 35)

1. (a) If $y = \log(ax + x^2)$, find y_n. 1

(b) If $y = (x + \sqrt{1 + x^2})^m$, then show that

$$(1 + x^2)y_2 + xy_1 - m^2 y = 0$$ 2

(c) Evaluate any *one* of the following : 3

(i) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

(ii) $\lim_{x \to 0} x^2 \log(x^2)$

(d) Find the radius of curvature at any point (x, y) for the curve $y = \log(\sin x)$. 4

Or

Derive the relation $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

2. (a) Choose the correct answer for the following : 1

If a function is continuous in a closed interval $[a, b]$, then

(i) it has a singular point

(ii) it is bounded

(iii) it is unbounded

(iv) it is infinite at some point of (a, b)

(b) Give an example of a function which is continuous, but not differentiable. 1

(c) If two functions have equal derivatives at all points in an interval (a, b), then write by what they differ. 1

(d) Write geometrical interpretation of Lagrange's mean value theorem. 2

(e) Prove that if a function f is continuous on a closed interval $[a, b]$, then it attains its bounds at least once in $[a, b]$. 5

Or

Show that $f(x) = x\tan^{-1}\left(\frac{1}{x}\right)$, for $x \neq 0$ and $f(0) = 0$, is not differentiable at $x = 0$.

3. (a) If $f = x\cos y$, find $\frac{\partial^2 f}{\partial x \partial y}$. 1

(b) If

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

4

Or

Verify Euler's theorem for the function

$$u = \frac{x - y}{x + y}$$

4\. (a) Choose the correct answer for the following : 1

If f_{xy} and f_{yx} are both continuous at (a, b), then

(i) $f_x(a, b) = f_y(a, b)$

(ii) $f_{xy}(a, b) = f_{yx}(a, b)$

(iii) $f_{xy}(a, b) \neq f_{yx}(a, b)$

(iv) $f_{xx}(a, b) = f_{yy}(a, b)$

(b) Define continuity of a function $f(x, y)$ at any point (a, b). 2

(c) Show that

$$\lim_{(x, y)\to(0, 0)} \frac{2xy^2}{x^2+y^2}$$

does not exist. 3

(d) Examine the equality of f_{xy} and f_{yx} for the function $f = x^3y + e^{xy^2}$. 4

Or

Determine the extreme value(s) of the function $f = y^2 + 4xy + 3x^2 + x^3$, if any.

GROUP—B

(Integral Calculus)

(*Marks* : 20)

5. (a) Write the condition, when

$$\int_0^{2a} f(x)\,dx = 0$$ 1

(b) Show that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$ 2

(c) Evaluate (any *one*) : 3

(i) $\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$

(ii) $\int_0^{\pi} x \sin x \cos^2 x\,dx$

(d) Obtain the reduction formula for

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx$$ 4

Or

Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$.

6. (a) Find the length of the arc of the parabola $y^2 = 4ax$ from the vertex to one extremity of the latus rectum. 5

Or

Find the length of the arc of the curve $x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$, from $\theta = 0$ to $\frac{\pi}{2}$.

(b) The circle $x^2 + y^2 = a^2$ revolves round the x-axis. Find the surface area. 5

Or

Show that the volume of a right circular cone of height h and base of radius a is

$$\frac{1}{3}\pi a^2 h$$

GROUP—C

(Riemann Integral)

(*Marks* : 25)

7. (a) Write the condition of Riemann integrability of a function. 1

(b) Give an example of a Riemann integrable function in $[a, b]$, which is not monotonic in $[a, b]$. 2

(c) Prove that every continuous function is Riemann integrable. 5

Or

If P_1 is a refinement of a partition P, then for a bounded function f, show that $L(P_1, f) \geq L(P, f)$.

8. (a) Choose the correct answer for the following : 1

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

(i) is not always valid

(ii) is always valid

(iii) f' is always bounded

(iv) None of the above

(b) Write the statement of first mean value theorem of integral calculus. 2

(c) Examine the Riemann integrability of the function

$$\begin{aligned} f(x) &= x\cos\tfrac{1}{x}, \quad x \neq 0 \\ &= 0 \qquad\quad ,\quad x = 0 \end{aligned}$$ 4

Or

Prove that if f and g are integrable on $[a, b]$ and g keeps the same sign over $[a, b]$, then there exists a number c lying between the bounds of f such that

$$\int_a^b fg\,dx = c\int_a^b g\,dx$$

9. (a) Write the reason(s) why

$$\int_1^\infty \frac{dx}{x^2}$$

is an improper integral. 1

(b) Write the statement of comparison test of convergence. 1

(c) Examine the convergence of

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$ 4

Or

Show that $\int_0^1 \frac{\log x}{\sqrt{x}}$ is convergent.

10. (a) Write the value of $\Gamma(1)$. 1

(b) Write $B(m, n)$ in terms of Gamma function. 1

(c) Find the value of $\Gamma\left(\frac{1}{2}\right)$. 2

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