Total No. of Printed Pages—8

3 SEM TDC MTH M 1

2016

(November)

MATHEMATICS

(Major)

Course : 301

[Analysis—I (Real Analysis)]

Full Marks : 80

Pass Marks : 32 (Backlog)/24 (2014 onwards)

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Differential Calculus)

(*Marks* : 35)

1. *(a)* Define curvature of a curve at any point. 1

(b) If $y = \frac{x^3}{x^2 - 1}$, then prove that for $n > 1$

$$y_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -n, & \text{if } n \text{ is odd} \end{cases}$$ 4

Or

Find :

$$\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

(c) Show that for the curve $by^2 = (x + a)^3$, the square of the subtangent varies as the subnormal. 2

(d) For any curve $r = f(\theta)$, show that

$$ds = \left[1 + \left(r\frac{dr}{d\theta}\right)^2\right]^{1/2} dr$$

3

2. (a) If a quadratic function is defined on $[a, b]$ by $f(x) = \alpha x^2 + \beta x + \gamma$, $\alpha \neq 0$, then find the real number c guaranteed by Lagrange's mean value theorem. 2

(b) State Darboux's theorem. 1

(c) Examine the validity of the hypothesis and conclusion of Rolle's theorem for the function

$$f(x) = (x - a)^m (x - b)^n; \quad x \in [a, b], m, n$$

being positive integers. 3

(d) State the condition under which a function can be expanded as a Maclaurin's series and hence obtain series expansion of $f(x) = e^{2x}$, $x \in \mathbb{R}$. 4

Or

State and prove Cauchy's mean value theorem.

3. (a) State Euler's theorem on homogeneous functions. 1

(b) If the functions

$$\frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial^2 u}{\partial x \partial y}$$

both exist for a particular set of values of x, y and one of them is continuous there, then show that

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

4

Or

If $f(x, y, z)$ admits of continuous partial derivatives and satisfies the relation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x, y, z)$$

where n is a positive integer, then prove that $f(x, y, z)$ is a homogenous function of degree n.

4. *(a)* Define functional determinant of the functions $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$. 1

(b) Show that

$$\lim_{(x,\ y)\to(0,\ 0)} \tan^{-1}\left(\frac{y}{x}\right)$$

does not exist. 2

(c) Discuss the nature of extreme value of

$$f(x) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

at (6, 0). 4

Or

Prove that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

is invariant for change of rectangular axes.

(d) Show that the function

$$f(x) = 2x^4 - 3x^2 y + y^2$$

has neither a maximum nor a minimum at (0, 0). 3

GROUP—B

(Integral Calculus)

(*Marks* : 20)

5. (a) Under what condition

$$\int_0^{2a} \phi(x)\,dx = 2\int_0^a \phi(x)\,dx?$$ 1

(b) Find the reduction formula for

$$\int \cos^n x\,dx$$ 3

(c) Evaluate : 3

$$\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\,d\theta$$

Or

$$\int_0^{\pi/2} \sin^6 x\,dx$$

(d) Prove that

$$\int_0^1 x^{3/2}(1-x)^{3/2}\,dx = \frac{3\pi}{128}$$ 3

6. (a) The arc length of the curve $y = f(x)$ lying between two points for which $x = a$ and $x = b$ $(b > a)$ is given by ____.

(Fill in the blank) 1

(b) Find the whole length of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$ 4

(c) Find the volume of the solid generated by revolution of the loop of the curve $y^2 = x^2(a - x)$ about the X-axis. 5

Or

The part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.

GROUP—C

(Riemann Integral)

(*Marks* : 25)

7. (a) Condition of continuity is necessary and sufficient for Riemann integrability.

(State True or False) 1

(b) Find the upper and lower Riemann integrals for the function f defined on $[0, 1]$ as follows :

$f(x) = (1 - x^2)^{1/2}$; if x is rational

and $f(x) = 1 - x$; if x is irrational

Hence show that f is not Riemann integral on $[0, 1]$. 3

(*Continued*)

(c) Prove that if a bounded function f, having a finite number of points of discontinuity on $[a, b]$, then it is R-integrable on $[a, b]$. 4

Or

Prove the necessary and sufficient condition for the integrability of a bounded function f is that to every $\varepsilon > 0$ there corresponds a partition P such that $U(P, f) - L(P, f) < \varepsilon$.

8. (a) A derivable function ϕ, if it exists, such that its derivative ϕ' is equal to a given function f is called _____.

(Fill in the blank) 1

(b) If f is bounded and integrable on $[a, b]$, then prove that there exists a number μ, lying between the bounds of f, such that $\int_a^b f(x)\,dx = \mu\,(b-a)$. 2

(c) If $\int_a^b f(x)\,dx$ and $\int_a^b \phi(x)\,dx$ both exist and $f(x)$ keeps the same sign throughout the interval $[a, b]$, then prove that there exists a number c such that

$$\int_a^b f(x)\,\phi(x)\,dx = c\int_a^b f(x)dx,\ m \le c \le M$$

where m and M bounds of ϕ on $[a, b]$. 4

Or

Show that $\lim\{I_n\}$ where

$$I_n = \int_0^n \frac{\sin nx}{x}\, dx, \; n \in I$$

exists and that the limit is equal to $\pi / 2$.

9. (a) Write the statement of Abel's test for the convergence of first and second kinds of integral of the product of two functions. 2

(b) Examine the convergence of the improper integral

$$\int_0^1 \frac{dx}{(1 - x^2)^{1/2}}$$

4

Or

Show that

$$\int_0^\infty x^{n-1} e^{-x} dx$$

is convergent if and only if $n > 0$.

10. Prove that

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p, q) = B(q, p)$$

4

★★★