3 SEM TDC MTH M 1

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(November)

MATHEMATICS

(Major)

Course : 301

[Analysis—I (Real Analysis)]

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Differential Calculus)

(*Marks* : 35)

1. (a) If $y = \log(2x+4)$, then write the value of y_n. 1

(b) Find the limit : 2

$$\underset{x\to 0}{\text{Lt}}\left[\frac{xe^x - \log(1+x)}{x^2}\right]$$

(c) Define subtangent. Show that the subtangent at any point of a parabola varies as the abscissa of the point of contact. 1+2=3

(d) If $y = \sin(a \sin^{-1} x)$, then prove that

$$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - a^2)y_n = 0$$

4

Or

Show that the radius of curvature at the origin of the conic $x^2 + y^2 + xy - y + x = 0$ is $\frac{1}{2\sqrt{2}}$.

2. (a) Give an example of a continuous function in a domain which has neither infimum nor supremum therein. 1

(b) If a function $f(x)$ satisfies the conditions of Lagrange's mean value theorem and also $f'(x) = 0$, $\forall\ x \in (a, b)$, then show that $f(x)$ is constant on $[a, b]$. 2

(c) If $a < b$, then prove that

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

4

Or

Using Maclaurin's theorem, expand $\sin x$ in an infinite series.

(d) Discuss the applicability of the Rolle's theorem for $f(x) = x^2 - 3x + 2$ in $[1, 2]$. 3

3. (a) Define homogeneous function of two variables. 1

(b) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$ 4

Or

If u is a homogeneous function, of degree n, of x and y, then show that

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 u = n(n-1)u$$

4. (a) State Young's theorem. 1

(b) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$$ 4

(c) If f and g are twice differentiable functions and $y = f(x + ct) + g(x - ct)$, then prove that $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$, c is a constant. 5

Or

Find the extreme values of

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

GROUP—B

(Integral Calculus)

(*Marks* : 20)

5. (a) Write the reduction formula for

$$\int_0^{\pi/4} \tan^n \theta \, d\theta$$ 1

(b) Prove that

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$$ 2

(c) Show that

$$\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2$$ 4

(d) Evaluate : 3

$$\int_0^{\pi/2} \cos^6 x \, dx$$

Or

Prove that

$$\int_0^{\pi/2} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{4}$$

6. (a) If α and β be the vectorial angles of A and B respectively, then write the formula for arc $\widehat{AB}$. 1

(b) Find the perimeter of the cardioid $r = a(1 + \cos\theta)$. 5

(c) Find the area of the surface of revolution formed by revolving the curve $r = 2a\cos\theta$ about the initial line. 4

Or

Find the volume of the solid generated by revolving the asteroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

about the x-axis.

GROUP—C

(Riemann Integral)

(*Marks* : 25)

7. (a) Define refinement of a partition. 1

(b) Show that the function f defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 2, & \text{if } x \text{ is irrational} \end{cases}$$

is not integrable on any interval. 2

(c) Prove that every continuous function is Riemann integrable. 5

Or

If

$$f(x) = \frac{1}{2^n}, \text{ when } \frac{1}{2^{n+1}} \le x \le \frac{1}{2^n} \; (n = 0, 1, 2, \ldots)$$
$$= 0, \text{ when } x = 0$$

then show that $f \in R[0, 1]$ although it has many points of discontinuity.

8. (a) State the fundamental theorem of integral calculus. 1

(b) If f is a continuous function on $[a, b]$, then show that there exists a number $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$ 3

(c) If a function f is bounded and integrable on $[a, b]$, then the function F defined as

$$F(x) = \int_a^x f(t)\,dt \quad a \le x \le b$$

is continuous on $[a, b]$. Prove it. 3

Or

Verify the mean value theorem for $f(x) = x$, $g(x) = e^x$ in the interval $[-1, 1]$.

9. (a) State the Dirichlet test for convergence of integral of a product. 1

(b) Test for convergence of

$$\int_0^\infty \frac{\sin x}{x}\, dx$$ 2

(c) Prove that $\int_1^\infty \frac{dx}{x^n}$ is convergent for $n > 1$. 3

10. Answer any *one* of the following : 4

(a) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(b) Prove that $\beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$.

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