3 SEM TDC STS M 1 (N/O)

2016

(November)

STATISTICS

(Major)

Course : 301

(Probability and Distribution—I)

(New Course)

Full Marks : 80

Pass Marks : 24

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct answer : 1×8=8

   (a) Two events are said to be independent if

      (i) each outcome has equal chance of occurrence

      (ii) there is no common point in between them

      (iii) both the events have only one point

      (iv) one does not affect the occurrence of the other

(b) If A_1, A_2 and A_3 are three mutually exclusive events, the probability of their union is equal to

(i) $P(A_1)P(A_2)P(A_3)$

(ii) $P(A_1)+P(A_2)+P(A_3)-P(A_1A_2A_3)$

(iii) $P(A_1)+P(A_2)+P(A_3)$

(iv) $P(A_1)P(A_2)+P(A_1)P(A_3)+P(A_2)P(A_3)$

(c) If A and B are two independent events, then $P(\overline{A}\cap\overline{B})$ is equal to

(i) $P(\overline{A})\,P(\overline{B})$

(ii) $1-P(A\cup B)$

(iii) $[1-P(A)]\,[1-P(B)]$

(iv) All of the above

(d) If $F(x)$ is a distribution function of a random variable, then

(i) $F(x)>1$

(ii) $F(x)<0$

(iii) $0\leq F(x)\leq 1$

(iv) None of the above

(e) If X is a random variable, then $E(e^{itX})$ is known as

(i) characteristic function

(ii) moment generating function

(iii) probability generating function

(iv) None of the above

(f) If X and Y are two random variables with means $\overline{X}$ and $\overline{Y}$ respectively, then the expression $E[(X-\overline{X})(Y-\overline{Y})]$ is called

(i) variance of X

(ii) variance of Y

(iii) covariance of X and Y

(iv) moments of X and Y

(g) If X_1 and X_2 are independent random variables, then the $M_{X_1+X_2}(t)$ is equal to

(i) $M_{X_1}(t)+M_{X_2}(t)$

(ii) $M_{X_1}(t)-M_{X_2}(t)$

(iii) $M_{X_1}(t)\ M_{X_2}(t)$

(iv) None of the above

(h) From a pack of 52 cards, two cards are drawn at random. The probability that one is an ace and the other is a king is

(i) 2/13

(ii) 8/663

(iii) 1/169

(iv) 16/169

2. Answer any *five* of the following :

(a) Define the statistical or empirical definition of probability. What are the differences between the classical and statistical definition of probability? 1+1=2

(b) State and prove the addition law of probability of two events A and B. 2

(c) Define marginal density function for a random variable X. 2

(d) Define mathematical expectation of a random variable. Prove that if X and Y are independent random variables, then $E(XY) = E(X)E(Y)$. 1+1=2 *(e)* If A, B and C are three arbitrary events, then find the expression for the events noted below, in the context of A, B and C : 1+1=2

(i) At least two occur

(ii) One and not more occurs

(f) Write at least two limitations of moment generating function. 2

3. Answer any *five* questions :

(a) If $B < A$, then prove that

(i) $P(A \cap \overline{B}) = P(A) - P(B)$

(ii) $P(B) \leq P(A)$ 2+2=4

(b) A problem in statistics is given to three students A, B and C whose chances of solving it are $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ respectively. What is the probability that the problem will be solved if all of them try independently? 4

(c) What is meant by distribution function of a random variable X? What are the common properties of a distribution? 2+2=4

(d) The joint probability density function of a two-dimensional random variable (X, Y) is given by

$$f(x, y) = 2;\ 0 < x < 1,\ 0 < y < x$$
$$= 0;\ \text{elsewhere}$$

(i) Find the marginal density function of X and Y.

(ii) Check the independence of X and Y. 2+2=4

(e) Prove that the moment generating function of the sum of a number of independent random variables is equal to the product of their respective moment generating functions. 4

(f) Joint distribution of a pair of random variables is given by the following table :

$X \rightarrow$ $Y \downarrow$	-1	0	1
0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
1	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
2	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$

Find the marginal distribution of X and Y. 4

4. (a) (i) What is meant by conditional probability? What are its properties? 1+2=3

(ii) State and prove the Bayes' theorem. 5

Or

(b) There are three machines producing 10000, 20000 and 30000 bullets per hour respectively. These machines are known to produce 5%, 4% and 2% defective bullets respectively. One bullet is taken at random from an hour's production of the three machines. What is the probability that it is defective? If the drawn bullet is defective, what is the probability that this was produced by the 2nd machine? 8

5. (a) Let X is a continuous random variable with probability density function,

$$\begin{aligned} f(x) &= ax;\ 0 \le x \le 1 \\ &= a;\ 1 \le x \le 2 \\ &= -ax + 3a;\ 2 \le x \le 3 \\ &= 0;\ \text{elsewhere} \end{aligned}$$

(i) Determine the value of a.

(ii) Compute $P(X \le 2 \cdot 5)$. 4+4=8

Or

(*b*) Let the random variable X assumes the value r with the probability law

$$P(x = r) = q^{r-1} p; \ r = 1, 2, 3 \cdots$$

Find the moment generating function and its mean and variance. 8

6. (*a*) Two random variables X and Y have the following joint density function :

$$f(x, y) = 2 - x - y; \ 0 \le x \le 1, \ 0 \le y \le 1$$
$$= 0, \text{ otherwise}$$

Find—

(*i*) marginal probability distributions of X and Y;

(*ii*) conditional density function of X / Y and Y / X;

(*iii*) $V(X)$ and $V(Y)$. 9

Or

(*b*) (*i*) Define the covariance between two variables in terms of X and Y. What is the covariance between two variables when X and Y are independent? 2+2=4

(*ii*) Let $X_1, X_2, \ldots, X_n$ be n random variables, then prove that

$$V\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 V(X_i) + 2\sum_{i}^{n}\sum_{j}^{n} a_i a_j \operatorname{cov}(X_i, X_j), \ i \ne j$$ 5

7. (a) What is meant by correlation? Explain how the limits of correlation lie between -1 to $+1$. Prove that correlation coefficient is independent of change of origine and scale. 2+2+4=8

Or

(b) What is meant by regression? Let (X, Y) be a two-dimensional random variable with $E(X) = \overline{X}$ and $E(Y) = \overline{Y}$, $V(X) = \sigma_X^2$ and $V(Y) = \sigma_Y^2$ and let $r = r(X, Y)$ be the correlation coefficient between X and Y. If the regression of Y on X is linear, then show that it can be represented by,

$$E(Y / X) = \overline{Y} + r\frac{\sigma_X}{\sigma_Y}(X - \overline{X})$$

2+6=8

8. Write short notes on any *three* of the following : 3×3=9

(a) Random variables

(b) Two lines of regression

(c) Mathematical expectation

(d) Probability mass function

(e) Cumulants

(Old Course)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct answer : 1×8=8

(*a*) Performing of a random experiment is called

(*i*) an event

(*ii*) a trial

(*iii*) a simple event

(*iv*) a composite event

(*b*) If a random variable X assumes the value 1 with probability P and O with probability $q = 1 - P$, then

(*i*) $E(X) = P$

(*ii*) $E(X) = nP$

(*iii*) $E(X) = q / P$

(*iv*) $E(X) nPq$

(c) The idea of posterior probabilities was introduced by

(i) Pascal

(ii) De Moivre

(iii) Thomas Bayes

(iv) Laplace

(d) A coin is tossed three times in succession, then the number of sample points in the sample space is

(i) 6

(ii) 8

(iii) 3

(iv) None of the above

(e) For two random variable X and Y with $E(X) = 2$, $E(Y) = 4$, $E(2X - 5Y)$ will be

(i) -16

(ii) -2

(iii) 24

(iv) 108

(f) The conditional probability density function of X, given $Y = y$ for a joint density $f_{XY}(x, y)$ can be found by the formula

(i) $f_{X/Y}(x/y) = \dfrac{f_{XY}(x, y)}{f_X(x)}$

(ii) $f_{X/Y}(x/y) = f_{Y/X}(y) f_{XY}(x, y)$

(iii) $f_{X/Y}(x/y) = \dfrac{f_{XY}(x, y)}{f_Y(y)}$

(iv) None of the above

(g) The probability generating function (p.g.f.) of the sum of a number of independent random variables (non-negative integral valued) is equal to

(i) 1

(ii) the product of their p.g.f.

(iii) the sum of their p.g.f.

(iv) None of the above

(h) If $E(Y/X)$ is the conditional expectation of Y, given $X = x$, then $E(XY)$ in terms of conditional expectation can be expressed as

(i) $E(XY) = E(X)E(Y/X)$

(ii) $E(XY) = E(Y)E(Y/X)$

(iii) $E(XY) = XE(Y/x)$

(iv) $E(XY) = E[XE(Y/X)]$

2. Answer the following questions : $2\times8=16$

(a) Define the classical definition of probability and mention its limitation.

(b) A, B and C are three arbitrary events. Find the expression for the events noted below :

(i) Both A and B, but not C occur

(ii) At least two occur

(iii) One and no more occur

(iv) Two and no move occur

(c) What is meant by conditional probability?

(d) What are the properties of distribution function?

(e) Define characteristics function and give its importance.

(f) Find the moment generating function of the random variable whose moments are given by $\mu_r' = \underline{|(r+1)}\,.2^r$.

(g) What is meant by marginal distribution and conditional distribution?

(h) If X is a random variable and a and b are constants, then show that—

$$V(aX+b)=a^2V(X)$$

3. (a) State and prove addition theorem of probability. 4

(b) A card is drawn from a well shuffled pack of cards. What is the probability that it is either a spade or an ace? 3

4. (a) Prove that for any two events A and B

$$P(A \cap B) \leq P(B) \leq P(A \cup B) \leq P(A) + P(B)$$ 4

(b) When are two events said to be independent and mutually exclusive? Prove that two independent events can not be mutually disjoint. 3

(c) A box contains 6 red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn, there are at least one ball of each colour. 3

5. (a) Define moment generating function of a random variable. A random variable X has the probability function

$$P(X = r) = q^{r-1} p \; ; \; r = 1, 2, 3 \cdots$$

Find the m.g.f. of X and hence its mean. 6

Or

(*b*) Define the probability generating function of a non-negative integral valued random variable. Establish its relationship with the moment generating function and cumulant generating function. If $P(S)$ is the probability generating function of the random variable X, find the generating functions for $\frac{X-a}{b}$, where a and b are constants. 6

6. One shot is fired from each of the three guns. A, B, C denote the events that the target is hit by the first, second and third guns respectively. If $P(A)=0\cdot5$, $P(B)=0\cdot6$ and $P(C)=0\cdot8$ and A, B, C are independent events, find the probability that (*i*) exactly one hit is registered and (*ii*) at least two hits are registered. 4

7. (*a*) Define mathematical expectation of a discrete and a continuous random variable. State its two important theorems. 3

(*b*) A box contains a white and b black balls. c balls are drawn. Find the expected value of the number of white balls drawn. 5

Or

(c) A coin is tossed until a head appears. What is the expectation of the number of tosses required? 5

8. (a) Given $f(x, y) = xe^{-x(y+1)}$; $x \geq 0$, $y \geq 0$. Find the regression curve of Y on X. 6

(b) Two random variables X and Y have the following joint probability density function :

$$f(x, y) = 2 - x - y \; ; \; 0 \leq x \leq 1, \; 0 \leq y \leq 1$$

Find—

(i) marginal probability density function of X and Y;

(ii) conditional density function;

(iii) $\text{var}(X)$ and $\text{var}(Y)$;

(iv) covariance between X and Y.

2+2+4+2=10

9. Define cumulant generating function. Show that the rth cumulant for the distribution

$$f(x) = Ke^{-Kx}, \; 0 < x < \infty, \; K \neq 0 \in R$$

is $\frac{1}{K^r} \underline{|(r-1)}$. 5

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